This chapter focuses on exponents and logarithms, along with applications of these crucial concepts.

Each positive number \( b \neq 1 \) leads to an exponential function \( b^x \). The inverse of this function is the logarithm base \( b \). Thus \( \log_b y = x \) means \( b^x = y \). We will see that the important algebraic properties of logarithms follow directly from the algebraic properties of exponents.

We will use exponents and logarithms to model radioactive decay, earthquake intensity, sound intensity, and star brightness. We will also see how functions with exponential growth describe population growth and compound interest.

Our approach to the magical number \( e \) and the natural logarithm lead to several important approximations that show the special properties of \( e \). These approximations demonstrate why the natural logarithm deserves its name.

This chapter concludes by relooking at exponential growth through the lens of our new knowledge of \( e \). We will see how \( e \) is used to model continuously compounded interest and continuous growth rates.
3.1 Logarithms as Inverses of Exponential Functions

LEARNING OBJECTIVES

By the end of this section you should be able to

- use exponential functions;
- evaluate logarithms in simple cases;
- use logarithms to find inverses of functions involving \( b^x \);
- compute the number of digits in a positive integer from its common logarithm.

Exponential Functions

We defined the meaning of a rational exponent in Section 2.3, but an expression such as \( 7^{\sqrt{2}} \) has not yet been defined. Nevertheless, the following example should make sense to you as the only reasonable (rational?) way to think of an irrational exponent.

Example 1

Find an approximation of \( 7^{\sqrt{2}} \).

Solution

Because \( \sqrt{2} \) is approximately 1.414, we expect that \( 7^{\sqrt{2}} \) should be approximately \( 7^{1.414} \) (which has been defined, because 1.414 is a rational number). A calculator shows that \( 7^{1.414} \approx 15.66638 \).

If we use a better approximation of \( \sqrt{2} \), then we should get a better approximation of \( 7^{\sqrt{2}} \). For example, 1.41421356 is a better rational approximation of \( \sqrt{2} \) than 1.414. A calculator shows that

\[
7^{1.41421356} = 15.67289,
\]

which turns out to be correct for the first five digits after the decimal point in the decimal expansion of \( 7^{\sqrt{2}} \).

We could continue this process by taking rational approximations as close as we wish to \( \sqrt{2} \), thus getting approximations as accurate as we wish to \( 7^{\sqrt{2}} \).

The example above gives the idea for defining the meaning of an irrational exponent:

Irrational exponent

Suppose \( b > 0 \) and \( x \) is an irrational number. Then \( b^x \) is the number that is approximated by numbers of the form \( b^r \) as \( r \) takes on rational values that approximate \( x \).

The definition of \( b^x \) above does not have the level of rigor expected of a mathematical definition, but the idea should be clear from the example above. A rigorous approach to this question would take us beyond material appropriate for a precalculus course. Thus we will rely on our intuitive sense of the loose definition given above.
The box below summarizes the key algebraic properties of exponents. These are the same properties we saw earlier, but now we have extended the meaning of exponents to a larger class of numbers.

**Algebraic properties of exponents**

Suppose $a$ and $b$ are positive numbers and $x$ and $y$ are real numbers. Then

\[
\begin{align*}
    b^x b^y &= b^{x+y}, \\
    (b^x)^y &= b^{xy}, \\
    a^x b^x &= (ab)^x, \\
    b^0 &= 1, \\
    \frac{a^x}{b^y} &= \left(\frac{a}{b}\right)^x.
\end{align*}
\]

Now that we have defined $b^x$ for every positive number $b$ and every real number $x$, we can define a function $f$ by $f(x) = b^x$. These functions (one for each number $b$) are sufficiently important to have their own name:

**Exponential function**

Suppose $b$ is a positive number, with $b \neq 1$. Then the exponential function with base $b$ is the function $f$ defined by

\[f(x) = b^x.\]

For example, taking $b = 2$, we have the exponential function $f$ with base 2 defined by $f(x) = 2^x$. The domain of this function is the set of real numbers, and the range is the set of positive numbers.

The potential base $b = 1$ is excluded from the definition of an exponential function because we do not want to make exceptions for this base. For example, it is convenient (and true) to say that the range of every exponential function is the set of positive numbers. But the function $f$ defined by $f(x) = 1^x$ has the property that $f(x) = 1$ for every real number $x$; thus the range of this function is the set {1} rather than the set of positive numbers. To exclude this kind of exception, we do not call this function $f$ an exponential function.

Be careful to distinguish the function $2^x$ from the function $x^2$. The graphs of these functions have different shapes. The function $g$ defined by $g(x) = x^2$ is not an exponential function. For an exponential function such as the function $f$ defined by $f(x) = 2^x$, the variable appears in the exponent.
Logarithms Base 2

Consider the exponential function \( f \) defined by \( f(x) = 2^x \). The table here gives the value of \( 2^x \) for some choices of \( x \). We now define a new function, called the logarithm base 2, that is the inverse of the exponential function \( 2^x \).

Each time \( x \) increases by 1, the value of \( 2^x \) doubles; this happens because \( 2^{x+1} = 2 \cdot 2^x \).

Logarithm base 2

Suppose \( y \) is a positive number.

- The logarithm base 2 of \( y \), denoted \( \log_2 y \), is defined to be the number \( x \) such that \( 2^x = y \).

Short version:

- \( \log_2 y = x \) means \( 2^x = y \).

For example, \( \log_2 8 = 3 \) because \( 2^3 = 8 \). Similarly, \( \log_2 \frac{1}{32} = -5 \) because \( 2^{-5} = \frac{1}{32} \).

EXAMPLE 2

Find a number \( t \) such that \( 2^{1/(t-8)} = 5 \).

**SOLUTION** The equation above is equivalent to the equation \( \log_2 5 = \frac{1}{t-8} \).

Solving this equation for \( t \) gives \( t = \frac{1}{\log_2 5} + 8 \).

The definition of \( \log_2 y \) as the number \( x \) such that \( 2^x = y \)

means that if \( f \) is the function defined by \( f(x) = 2^x \), then the inverse function of \( f \) is given by the formula \( f^{-1}(y) = \log_2 y \). Thus the table here giving some values of \( \log_2 y \) is obtained by interchanging the two columns of the earlier table giving the values of \( 2^x \), as always happens with a function and its inverse.

Expressions such as \( \log_2 0 \) and \( \log_2 (-1) \) make no sense because there does not exist a number \( x \) such that \( 2^x = 0 \), nor does there exist a number \( x \) such that \( 2^x = -1 \).

The figure here shows part of the graph of \( \log_2 x \). Because the function \( \log_2 x \) is the inverse of the function \( 2^x \), flipping the graph of \( 2^x \) across the line \( y = x \) gives the graph of \( \log_2 x \).

Note that \( \log_2 \) is a function; thus \( \log_2(y) \) might be a better notation than \( \log_2 y \). In an expression such as

\[ \frac{\log_2 15}{\log_2 5} \]

we cannot cancel \( \log_2 \) in the numerator and denominator, just as we cannot cancel a function \( f \) in the numerator and denominator of \( \frac{f(15)}{f(3)} \). Similarly, the expression above is not equal to \( \log_2 3 \), just as \( \frac{f(15)}{f(3)} \) is usually not equal to \( f(3) \).
Logarithms with Any Base

We now take up the topic of defining logarithms with bases other than 2. No new ideas are needed for this more general situation—we simply replace 2 by a positive number \( b \neq 1 \). Here is the formal definition:

**Logarithm**

Suppose \( b \) and \( y \) are positive numbers, with \( b \neq 1 \).

- The **logarithm** base \( b \) of \( y \), denoted \( \log_b y \), is defined to be the number \( x \) such that \( b^x = y \).

**Short version:**

- \( \log_b y = x \) means \( b^x = y \).

---

**EXAMPLE 3**

(a) Evaluate \( \log_{10} 1000 \).

(b) Evaluate \( \log_7 49 \).

(c) Evaluate \( \log_3 \frac{1}{81} \).

**SOLUTION**

(a) Because \( 10^3 = 1000 \), we have \( \log_{10} 1000 = 3 \).

(b) Because \( 7^2 = 49 \), we have \( \log_7 49 = 2 \).

(c) Because \( 3^{-4} = \frac{1}{81} \), we have \( \log_3 \frac{1}{81} = -4 \).

Two important identities follow immediately from the definition:

**The logarithm of 1 and the logarithm of the base**

If \( b \) is a positive number with \( b \neq 1 \), then

- \( \log_b 1 = 0 \);
- \( \log_b b = 1 \).

The first identity holds because \( b^0 = 1 \); the second holds because \( b^1 = b \).

The definition of \( \log_b y \) as the number \( x \) such that \( b^x = y \) has the following consequence:

**Logarithm as an inverse function**

Suppose \( b \) is a positive number with \( b \neq 1 \) and \( f \) is the exponential function defined by \( f(x) = b^x \). Then the inverse function of \( f \) is given by the formula

\[
f^{-1}(y) = \log_b y.
\]"
Because a function and its inverse interchange domains and ranges, the domain of the function \( f^{-1} \) defined by \( f^{-1}(y) = \log_b y \) is the set of positive numbers, and the range of this function is the set of real numbers.

**EXAMPLE 4**

Suppose \( f \) is the function defined by \( f(x) = 3 \cdot 5^{x-7} \). Find a formula for \( f^{-1} \).

**SOLUTION**  
To find a formula for \( f^{-1} (y) \), we solve the equation \( 3 \cdot 5^{x-7} = y \) for \( x \). Dividing by 3, we have \( 5^{x-7} = \frac{y}{3} \). Thus \( x - 7 = \log_5 \frac{y}{3} \), which implies that \( x = 7 + \log_5 \frac{y}{3} \). Hence

\[
 f^{-1}(y) = 7 + \log_5 \frac{y}{3}.
\]

Because the function \( \log_b x \) is the inverse of the function \( b^x \), flipping the graph of \( b^x \) across the line \( y = x \) gives the graph of \( \log_b x \). If \( b > 1 \) then \( \log_b x \) is an increasing function (because \( b^x \) is an increasing function). As we will see in the next section, the shape of the graph of \( \log_b x \) is similar to the shape of the graph of \( \log_2 x \) obtained earlier.

The definition of logarithm implies the two equations displayed below. Be sure that you are comfortable with these equations and understand why they hold. Note that if \( f \) is defined by \( f(x) = b^x \), then \( f^{-1}(y) = \log_b y \). The equations below could then be written in the form \( (f \circ f^{-1})(y) = y \) and \( (f^{-1} \circ f)(x) = x \), which are equations that always hold for a function and its inverse.

### Inverse properties of logarithms

If \( b \) and \( y \) are positive numbers, with \( b \neq 1 \), and \( x \) is a real number, then

- \( b^{\log_b y} = y \);
- \( \log_b b^x = x \).

### Common Logarithms and the Number of Digits

In applications of logarithms, the most commonly used values for the base are 10, 2, and the number \( e \) (which we will discuss in Section 3.5). The use of a logarithm with base 10 is so frequent that it gets a special name:

#### Common logarithm

- The logarithm base 10 is called the **common logarithm**.

- To simplify notation, sometimes logarithms base 10 are written without the base. If no base is displayed, then the base is assumed to be 10. In other words,

\[
\log y = \log_{10} y.
\]

Thus, for example, \( \log 10000 = 4 \) (because \( 10^4 = 10000 \)) and \( \log \frac{1}{100} = -2 \) (because \( 10^{-2} = \frac{1}{100} \)). If your calculator has a button labeled “log”, then it will compute the logarithm base 10, which is often just called the logarithm.
Note that $10^1$ is a two-digit number, $10^2$ is a three-digit number, $10^3$ is a four-digit number, and more generally, $10^{n-1}$ is an $n$-digit number. Thus the integers with $n$ digits are the integers in the interval $[10^{n-1}, 10^n)$. Because $\log 10^{n-1} = n - 1$ and $\log 10^n = n$, this implies that an $n$-digit positive integer has a logarithm in the interval $[n - 1, n)$.

**Digits and logarithms**

The logarithm of an $n$-digit positive integer is in the interval $[n - 1, n)$.

The conclusion above is often useful in making estimates. For example, without using a calculator we can see that the number 123456789, which has nine digits, has a logarithm between 8 and 9 (the actual value is about 8.09).

The next example shows how to use the conclusion above to determine the number of digits in a number from its logarithm.

**Example 5**

Always round up the logarithm of a number to determine the number of digits. Here $\log M \approx 73.1$ is rounded up to show that $M$ has 74 digits.

**EXERCISES**

For Exercises 1–6, evaluate the indicated quantities assuming that $f$ and $g$ are the functions defined by $f(x) = 2^x$ and $g(x) = \frac{x + 1}{x + 2}$.

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>$(f \circ g)(-1)$</td>
</tr>
<tr>
<td>2</td>
<td>$(g \circ f)(0)$</td>
</tr>
<tr>
<td>3</td>
<td>$(f \circ g)(0)$</td>
</tr>
<tr>
<td>4</td>
<td>$(g \circ f)(\frac{3}{2})$</td>
</tr>
<tr>
<td>5</td>
<td>$(f \circ f)(\frac{1}{2})$</td>
</tr>
<tr>
<td>6</td>
<td>$(f \circ f)(\frac{1}{2})$</td>
</tr>
</tbody>
</table>

For Exercises 7–8, find a formula for $f \circ g$ given the indicated functions $f$ and $g$.

7. $f(x) = 5x\sqrt{2}$, $g(x) = x^{\sqrt{3}}$

8. $f(x) = 7x\sqrt{12}$, $g(x) = x^{\sqrt{3}}$

For Exercises 9–24, evaluate the indicated expression. Do not use a calculator for these exercises.

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<table>
<thead>
<tr>
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<tbody>
<tr>
<td>9</td>
<td>$\log_2 64$</td>
</tr>
<tr>
<td>10</td>
<td>$\log_2 1024$</td>
</tr>
<tr>
<td>11</td>
<td>$\log_2 \frac{1}{128}$</td>
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<tr>
<td>12</td>
<td>$\log_2 \frac{1}{278}$</td>
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<tr>
<td>13</td>
<td>$\log_4 2$</td>
</tr>
<tr>
<td>14</td>
<td>$\log_8 2$</td>
</tr>
<tr>
<td>15</td>
<td>$\log_4 8$</td>
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<tr>
<td>16</td>
<td>$\log_8 128$</td>
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<tr>
<td>17</td>
<td>$\log_{10} 10000$</td>
</tr>
<tr>
<td>18</td>
<td>$\log_{10} \frac{1}{1000}$</td>
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<tr>
<td>19</td>
<td>$\log\sqrt{1000}$</td>
</tr>
<tr>
<td>20</td>
<td>$\log_2 \frac{1}{10000}$</td>
</tr>
<tr>
<td>21</td>
<td>$\log_2 8^{3.1}$</td>
</tr>
<tr>
<td>22</td>
<td>$\log_8 26.3$</td>
</tr>
<tr>
<td>23</td>
<td>$\log_{16} 32$</td>
</tr>
<tr>
<td>24</td>
<td>$\log_{27} 81$</td>
</tr>
<tr>
<td>25</td>
<td>Find a number $y$ such that $\log_2 y = 7$.</td>
</tr>
<tr>
<td>26</td>
<td>Find a number $t$ such that $\log_2 t = 8$.</td>
</tr>
<tr>
<td>27</td>
<td>Find a number $y$ such that $\log_2 y = -5$.</td>
</tr>
<tr>
<td>28</td>
<td>Find a number $t$ such that $\log_2 t = -9$.</td>
</tr>
</tbody>
</table>

For Exercises 29–36, find a number $b$ such that the indicated equality holds.

<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>29</td>
<td>$\log_b 64 = 1$</td>
</tr>
<tr>
<td>30</td>
<td>$\log_b 64 = 2$</td>
</tr>
<tr>
<td>31</td>
<td>$\log_b 64 = 3$</td>
</tr>
<tr>
<td>32</td>
<td>$\log_b 64 = 6$</td>
</tr>
<tr>
<td>33</td>
<td>$\log_b 64 = 12$</td>
</tr>
<tr>
<td>34</td>
<td>$\log_b 64 = 18$</td>
</tr>
<tr>
<td>35</td>
<td>$\log_b 64 = \frac{3}{2}$</td>
</tr>
<tr>
<td>36</td>
<td>$\log_b 64 = \frac{6}{3}$</td>
</tr>
</tbody>
</table>
For Exercises 37–48, find all numbers x such that the indicated equation holds.

37 \( \log |x| = 2 \)  
38 \( \log |x| = 3 \)  
39 \( |\log x| = 2 \)  
40 \( |\log x| = 3 \)  
41 \( \log_3(5x + 1) = 2 \)  
42 \( \log_4(3x + 1) = -2 \)  
43 \( 13 = 10^{2x} \)

For Exercises 49–66, find a formula for the inverse function \( f^{-1} \) of the indicated function \( f \).

49 \( f(x) = 3^x \)  
50 \( f(x) = 4.7^x \)  
51 \( f(x) = 2^{x-5} \)  
52 \( f(x) = 9^{x+6} \)  
53 \( f(x) = 6^x + 7 \)  
54 \( f(x) = 5^x - 3 \)  
55 \( f(x) = 4 \cdot 5^x \)  
56 \( f(x) = 8 \cdot 7^x \)  
57 \( f(x) = 2 \cdot 9^x + 1 \)

58 \( f(x) = 3 \cdot 4^x - 5 \)  
59 \( f(x) = \log_8 x \)  
60 \( f(x) = \log_3 x \)  
61 \( f(x) = \log_4 (3x + 1) \)  
62 \( f(x) = \log_7 (2x - 9) \)  
63 \( f(x) = 5 + 3 \log_6 (2x + 1) \)  
64 \( f(x) = 8 + 9 \log_2 (4x - 7) \)  
65 \( f(x) = \log_8 13 \)  
66 \( f(x) = \log_{5x} 6 \)

For Exercises 67–74, find a formula for \( (f \circ g)(x) \) assuming that \( f \) and \( g \) are the indicated functions.

67 \( f(x) = \log_6 x \) and \( g(x) = 6^{3x} \)  
68 \( f(x) = \log_5 x \) and \( g(x) = 5^{3+2x} \)  
69 \( f(x) = 6^{3x} \) and \( g(x) = \log_6 x \)  
70 \( f(x) = 5^{3+2x} \) and \( g(x) = \log_5 x \)  
71 \( f(x) = \log_4 4 \) and \( g(x) = 10^x \)  
72 \( f(x) = \log_{2x} 7 \) and \( g(x) = 10^x \)  
73 \( f(x) = \log_4 4 \) and \( g(x) = 100^x \)  
74 \( f(x) = \log_{2x} 7 \) and \( g(x) = 100^x \)

75 Find a number \( n \) such that \( \log_3 (\log_5 n) = 1 \).

76 Find a number \( n \) such that \( \log_3 (\log_2 n) = 2 \).

77 Find a number \( m \) such that \( \log_7 (\log_8 m) = 2 \).

78 Find a number \( m \) such that \( \log_5 (\log_6 m) = 3 \).

79 Suppose \( N \) is a positive integer such that \( \log N \approx 35.4 \). How many digits does \( N \) have?

80 Suppose \( k \) is a positive integer such that \( \log k \approx 83.2 \). How many digits does \( k \) have?

PROBLEMS

Some problems require considerably more thought than the exercises.

81 Show that \( (3\sqrt{2})^2 = 9 \).

82 Give an example of three irrational numbers \( x, y, \) and \( z \) such that \( (x^y)^z \) is a rational number.

83 Is the function \( f \) defined by \( f(x) = 2^x \) for every real number \( x \) an even function, an odd function, or neither?

84 Suppose \( f(x) = 8^x \) and \( g(x) = 2^x \). Explain why the graph of \( g \) can be obtained by horizontally stretching the graph of \( f \) by a factor of 3.

85 Suppose \( f(x) = 2^x \). Explain why shifting the graph of \( f \) left 3 units produces the same graph as vertically stretching the graph of \( f \) by a factor of 8.

86 Explain why there does not exist a polynomial \( p \) such that \( p(x) = 2^x \) for every real number \( x \).
[Hint: Consider the behavior of \( p(x) \) and \( 2^x \) for \( x \) near \( -\infty \).]

87 Explain why there does not exist a rational function \( r \) such that \( r(x) = 2^x \) for every real number \( x \).
[Hint: Consider the behavior of \( r(x) \) and \( 2^x \) for \( x \) near \( \pm\infty \).]

88 Explain why \( \log_5 \sqrt{5} = \frac{1}{2} \).

89 Explain why \( \log_4 100 \) is between 4 and 5.

90 Explain why \( \log_{40} 3 \) is between \( \frac{1}{4} \) and \( \frac{1}{3} \).

91 Show that \( \log_3 5 \) is an irrational number.
[Hint: Use proof by contradiction: Assume \( \log_3 5 \) is equal to a rational number \( \frac{m}{n} \); write out what this means, and think about even and odd numbers.]

92 Show that \( \log 2 \) is irrational.

93 Explain why logarithms with a negative base are not defined.

94 Give the coordinates of three distinct points on the graph of the function \( f \) defined by \( f(x) = \log_3 x \).

95 Give the coordinates of three distinct points on the graph of the function \( g \) defined by \( g(b) = \log_b 4 \).

96 Suppose \( g(b) = \log_b 5 \), where the domain of \( g \) is the interval \((1, \infty)\). Is \( g \) an increasing function or a decreasing function?
WORKED-OUT SOLUTIONS to Odd-Numbered Exercises

Do not read these worked-out solutions before attempting to do the exercises yourself. Otherwise you may mimic the techniques shown here without understanding the ideas.

For Exercises 1–6, evaluate the indicated quantities assuming that \( f \) and \( g \) are the functions defined by
\[
f(x) = 2^x \quad \text{and} \quad g(x) = \frac{x + 1}{x + 2}.
\]

1. \((f \circ g)(−1)\)
   \[
   (f \circ g)(−1) = f(g(−1)) = f(0) = 2^0 = 1
   \]

SOLUTION

2. \((f \circ g)(0)\)
   \[
   (f \circ g)(0) = f(g(0)) = f(\frac{1}{2}) = 2^{1/2} \approx 1.414
   \]

SOLUTION

3. \((f \circ f)(\frac{1}{2})\)
   \[
   (f \circ f)(\frac{1}{2}) = f(f(\frac{1}{2})) = f(2^{1/2})
   \]
   \[
   \approx f(1.41421)
   \]
   \[
   = 2^{1.41421}
   \]
   \[
   \approx 2.66514
   \]

For Exercises 7–8, find a formula for \( f \circ g \) given the indicated functions \( f \) and \( g \).

7. \( f(x) = 5x\sqrt{2}, \ g(x) = x\sqrt{3} \)

SOLUTION

\[
(f \circ g)(x) = f(g(x)) = f(x\sqrt{3})
\]
\[
= 5(x\sqrt{3})^{\sqrt{2}} = 5x^{\sqrt{18}} = 5x^4
\]

For Exercises 9–24, evaluate the indicated expression. Do not use a calculator for these exercises.

9. \( \log_2 64 \)

SOLUTION

If we let \( x = \log_2 64 \), then \( x \) is the number such that
\[
64 = 2^x.
\]
Because \( 64 = 2^6 \), we see that \( x = 6 \). Thus \( \log_2 64 = 6 \).

11. \( \log_2 \frac{1}{128} \)

SOLUTION

If we let \( x = \log_2 \frac{1}{128} \), then \( x \) is the number such that
\[
\frac{1}{128} = 2^x.
\]
Because \( \frac{1}{128} = \frac{1}{2^7} = 2^{-7} \), we see that \( x = -7 \). Thus \( \log_2 \frac{1}{128} = -7 \).

13. \( \log_2 2 \)

SOLUTION

Because \( 2 = 4^{1/2} \), we have \( \log_4 2 = \frac{1}{2} \).

15. \( \log_4 8 \)

SOLUTION

Because \( 8 = 2 \cdot 4 = 4^{1/2} \cdot 4 = 4^{3/2} \), we have \( \log_4 8 = \frac{3}{2} \).

17. \( \log 10000 \)

SOLUTION

\[
\log 10000 = \log 10^4
\]
\[
= 4
\]

19. \( \log \sqrt{1000} \)

SOLUTION

\[
\log \sqrt{1000} = \log 1000^{1/2}
\]
\[
= \log (10^3)^{1/2}
\]
\[
= \log 10^{3/2}
\]
\[
= \frac{3}{2}
\]

21. \( \log_2 8^{3.1} \)

SOLUTION

\[
\log_2 8^{3.1} = \log_2 (2^3)^{3.1}
\]
\[
= \log_2 2^{9.3}
\]
\[
= 9.3
\]

23. \( \log_{16} 32 \)
25 Find a number \( y \) such that \( \log_2 y = 7 \).

**Solution** The equation \( \log_2 y = 7 \) implies that
\[
y = 2^7 = 128.
\]

27 Find a number \( y \) such that \( \log_2 y = -5 \).

**Solution** The equation \( \log_2 y = -5 \) implies that
\[
y = 2^{-5} = \frac{1}{32}.
\]

For Exercises 29–36, find a number \( b \) such that the indicated equality holds.

29 \( \log_b 64 = 1 \)

**Solution** The equation \( \log_b 64 = 1 \) implies that
\[
b^1 = 64.
\]
Thus \( b = 64 \).

31 \( \log_b 64 = 3 \)

**Solution** The equation \( \log_b 64 = 3 \) implies that
\[
b^3 = 64.
\]
Because \( 4^3 = 64 \), this implies that \( b = 4 \).

33 \( \log_b 64 = 12 \)

**Solution** The equation \( \log_b 64 = 12 \) implies that
\[
b^{12} = 64.
\]
Thus
\[
b = 64^{1/12} = (2^6)^{1/12} = 2^{6/12} = 2^{1/2} = \sqrt{2}.
\]  

35 \( \log_b 64 = \frac{3}{2} \)

**Solution** The equation \( \log_b 64 = \frac{3}{2} \) implies that
\[
b^{3/2} = 64.
\]
Raising both sides of this equation to the \( 2/3 \) power, we get
\[
b = 64^{2/3} = (2^6)^{2/3} = 2^4 = 16.
\]

For Exercises 37–48, find all numbers \( x \) such that the indicated equation holds.

37 \( \log |x| = 2 \)

**Solution** The equation \( \log |x| = 2 \) is equivalent to the equation
\[
|x| = 10^2 = 100.
\]
Thus the two values of \( x \) satisfying this equation are \( x = 100 \) and \( x = -100 \).

39 \( |\log x| = 2 \)

**Solution** The equation \( |\log x| = 2 \) means that \( \log x = 2 \) or \( \log x = -2 \), which means that
\[
x = 10^2 = 100 \quad \text{or} \quad x = 10^{-2} = \frac{1}{100}.
\]

41 \( \log_3 (5x + 1) = 2 \)

**Solution** The equation \( \log_3 (5x + 1) = 2 \) implies that
\[
5x + 1 = 3^2 = 9.
\]
Thus \( 5x = 8 \), which implies that \( x = \frac{8}{5} \).

43 \( 13 = 10^{2x} \)

**Solution** The equation \( 13 = 10^{2x} \) implies that \( 2x = \log 13 \). Thus \( x = \frac{\log 13}{2} \), which is approximately 0.557.

45 \( \frac{10^x + 1}{10^x + 2} = 0.8 \)

**Solution** Multiplying both sides of the equation above by \( 10^x + 2 \), we get
\[
10^x + 1 = 0.8 \cdot 10^x + 1.6.
\]
Solving this equation for \( 10^x \) gives \( 10^x = 3 \), which means that \( x = \log 3 \approx 0.477121 \).
47 \[ 10^{2x} + 10^x = 12 \]

**SOLUTION** Note that \( 10^{2x} = (10^x)^2 \). This suggests that we let \( y = 10^x \). Then the equation above can be rewritten as
\[ y^2 + y - 12 = 0. \]

The solutions to this equation (which can be found either by using the quadratic formula or by factoring) are \( y = -4 \) and \( y = 3 \). Thus \( 10^x = -4 \) or \( 10^x = 3 \). However, there is no real number \( x \) such that \( 10^x = -4 \) (because \( 10^x \) is positive for every real number \( x \)), and thus we must have \( 10^x = 3 \). Thus \( x = \log 3 = 0.477121 \).

**For Exercises 49–66, find a formula for the inverse function \( f^{-1} \) of the indicated function \( f \).**

**49** \( f(x) = 3^x \)

**SOLUTION** By definition of the logarithm, the inverse of \( f \) is the function \( f^{-1} \) defined by
\[ f^{-1}(y) = \log_3 y. \]

**51** \( f(x) = 2^{x-5} \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation \( 2^{x-5} = y \) for \( x \). This equation means that \( x - 5 = \log_2 y \). Thus \( x = 5 + \log_2 y \). Hence
\[ f^{-1}(y) = 5 + \log_2 y. \]

**53** \( f(x) = 6^x + 7 \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation \( 6^x + 7 = y \) for \( x \). Subtract 7 from both sides, getting \( 6^x = y - 7 \). This equation means that \( x = \log_6 (y - 7) \). Hence
\[ f^{-1}(y) = \log_6 (y - 7). \]

**55** \( f(x) = 4 \cdot 5^x \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation \( 4 \cdot 5^x = y \) for \( x \). Divide both sides by 4, getting \( 5^x = \frac{y}{4} \). This equation means that \( x = \log_5 \frac{y}{4} \). Hence
\[ f^{-1}(y) = \log_5 \frac{y}{4}. \]

**57** \( f(x) = 2 \cdot 9^x + 1 \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation \( 2 \cdot 9^x + 1 = y \) for \( x \). Subtract 1 from both sides, then divide both sides by 2, getting \( 9^x = \frac{y - 1}{2} \). This equation means that \( x = \log_9 \frac{y - 1}{2} \). Hence
\[ f^{-1}(y) = \log_9 \frac{y - 1}{2}. \]

**59** \( f(x) = \log_8 x \)

**SOLUTION** By the definition of the logarithm, the inverse of \( f \) is the function \( f^{-1} \) defined by
\[ f^{-1}(y) = 8^y. \]

**61** \( f(x) = \log_4 (3x + 1) \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation \( \log_4 (3x + 1) = y \) for \( x \). This equation means that \( 3x + 1 = 4^y \). Solving for \( x \), we get \( x = \frac{4^y - 1}{3} \). Hence
\[ f^{-1}(y) = \frac{4^y - 1}{3}. \]

**63** \( f(x) = 5 + 3 \log_6 (2x + 1) \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation
\[ 5 + 3 \log_6 (2x + 1) = y \]
for \( x \). Subtracting 5 from both sides and then dividing by 3 gives
\[ \log_6 (2x + 1) = \frac{y - 5}{3}. \]
This equation means that \( 2x + 1 = 6^{(y - 5)/3} \). Solving for \( x \), we get \( x = \frac{6^{(y - 5)/3} - 1}{2} \). Hence
\[ f^{-1}(y) = \frac{6^{(y - 5)/3} - 1}{2}. \]

**65** \( f(x) = \log_x 13 \)

**SOLUTION** To find a formula for \( f^{-1}(y) \), we solve the equation \( \log_x 13 = y \) for \( x \). This equation means that \( x^y = 13 \). Raising both sides to the power \( \frac{1}{y} \), we get \( x = 13^{1/y} \). Hence
\[ f^{-1}(y) = 13^{1/y}. \]

**For Exercises 67–74, find a formula for \( (f \circ g)(x) \) assuming that \( f \) and \( g \) are the indicated functions.**

**67** \( f(x) = \log_6 x \) and \( g(x) = 6^{3x} \)

**SOLUTION**
\[ (f \circ g)(x) = f(g(x)) = f(6^{3x}) = \log_6 6^{3x} = 3x \]
69 \( f(x) = 6^{3x} \) and \( g(x) = \log_6 x \)

**SOLUTION**

\[
(f \circ g)(x) = f(g(x)) = f(\log_6 x) = 6^{3\log_6 x} = (6^{\log_6 x})^3 = x^3
\]

71 \( f(x) = \log_4 x \) and \( g(x) = 10^x \)

**SOLUTION**

\[
(f \circ g)(x) = f(g(x)) = f(10^x) = \log_{10^x} 4 = \frac{\log 4}{\log 10^x} = \frac{\log 4}{2x}
\]

73 \( f(x) = \log_4 x \) and \( g(x) = 100^x \)

**SOLUTION**

\[
(f \circ g)(x) = f(g(x)) = f(100^x) = \log_{100^x} 4
\]

75 Find a number \( n \) such that \( \log_3 (\log_5 n) = 1 \).

**SOLUTION** The equation \( \log_3 (\log_5 n) = 1 \) implies that \( \log_5 n = 3 \), which implies that \( n = 5^3 = 125 \).

77 Find a number \( m \) such that \( \log_7 (\log_8 m) = 2 \).

**SOLUTION** The equation \( \log_7 (\log_8 m) = 2 \) implies that \( \log_8 m = 7^2 = 49 \).

The equation above now implies that \( m = 8^{49} \).

79 Suppose \( N \) is a positive integer such that \( \log N \approx 35.4 \). How many digits does \( N \) have?

**SOLUTION** Because 35.4 is in the interval \([35, 36)\), we can conclude that \( N \) is a 36-digit number.

---

The graphs of \( \log_2 x \) (blue), \( \log_3 x \) (red), \( \log_4 x \) (green), \( \log_5 x \) (orange), and \( \log_6 x \) (purple) on the interval \([0.1, 100]\).
### Applications of the Power Rule for Logarithms

#### Learning Objectives
By the end of this section you should be able to
- apply the formula for the logarithm of a power;
- model radioactive decay using half-life;
- apply the change-of-base formula for logarithms.

#### Logarithm of a Power
Logarithms convert powers to products. To see this, suppose $b$ and $y$ are positive numbers, with $b \neq 1$, and $t$ is a real number. Let

$$x = \log_b y.$$ 

Then the definition of logarithm base $b$ implies

$$b^x = y.$$ 

Raise both sides of this equation to the power $t$ and use the identity $(b^x)^t = b^{tx}$ to get

$$b^{tx} = y^t.$$ 

Again using the definition of logarithm base $b$, the equation above implies

$$\log_b (y^t) = tx$$

$$= t \log_b y.$$ 

Thus we have the following formula for the logarithm of a power:

**Logarithm of a power**

If $b$ and $y$ are positive numbers, with $b \neq 1$, and $t$ is a real number, then

$$\log_b (y^t) = t \log_b y.$$ 

The next example shows a nice application of the formula above.

How many digits does $3^{5000}$ have?

**Solution** We can answer this question by evaluating the common logarithm of $3^{5000}$. Using the formula for the logarithm of a power and a calculator, we see that

$$\log(3^{5000}) = 5000 \log 3 \approx 2385.61.$$ 

Thus $3^{5000}$ has 2386 digits.
Before calculators and computers existed, books of common logarithm tables were frequently used to compute powers of numbers. As an example of how this worked, consider the problem of evaluating $1.7^{3.7}$. The key to performing this calculation is the formula

$$\log(1.7^{3.7}) = 3.7 \log 1.7.$$ 

Let’s assume we have a book that gives the logarithms of the numbers from 1 to 10 in increments of 0.001, meaning that the book gives the logarithms of 1.001, 1.002, 1.003, and so on.

The idea is first to compute the right side of the equation above. To do that, we would look in the book of logarithms, getting $\log 1.7 \approx 0.230449$. Multiplying the last number by 3.7, we would conclude that the right side of the equation above is approximately 0.852661. Thus, according to the equation above,

$$\log(1.7^{3.7}) \approx 0.852661.$$ 

Hence we can evaluate $1.7^{3.7}$ by finding a number whose logarithm equals 0.852661. To do this, we would look through our book of logarithms and find that the closest match is provided by the entry showing that $\log 7.123 \approx 0.852663$. Thus

$$1.7^{3.7} \approx 7.123.$$ 

Logarithms are rarely used today directly by humans for computations such as evaluating $1.7^{3.7}$. However, logarithms are used by your calculator for such computations. Logarithms also have important uses in calculus and several other branches of mathematics. Furthermore, logarithms have several practical uses, as we will soon see.

**Radioactive Decay and Half-Life**

Scientists have observed that starting with a large sample of radon atoms, after 92 hours one-half of the radon atoms will decay into polonium. After another 92 hours, one-half of the remaining radon atoms will also decay into polonium. In other words, after 184 hours, only one-fourth of the original radon atoms will be left. After another 92 hours, one-half of those remaining one-fourth of the original atoms will decay into polonium, leaving only one-eighth of the original radon atoms after 276 hours.

After $t$ hours, the number of radon atoms will be reduced by half $t/92$ times. Thus after $t$ hours, the number of radon atoms left will equal the original number of radon atoms divided by $2^{t/92}$. Here $t$ need not be an integer multiple of 92. For example, after five hours the original number of radon atoms will be divided by $2^{5/92}$. Because

$$\frac{1}{2^{5/92}} \approx 0.963,$$

this means that after five hours a sample of radon will contain 96.3% of the original number of radon atoms.

Because half the atoms in any sample of radon will decay to polonium in 92 hours, we say that radon has a **half-life** of 92 hours. Some radon atoms exist for less than 92 hours; some radon atoms exist for much longer than 92 hours.
The half-life of any radioactive isotope is the length of time it takes for half the atoms in a large sample of the isotope to decay. The table here gives the approximate half-life for several radioactive isotopes (the isotope number shown after the name of each element gives the total number of protons and neutrons in each atom of the isotope).

<table>
<thead>
<tr>
<th>isotope</th>
<th>half-life</th>
</tr>
</thead>
<tbody>
<tr>
<td>neon-18</td>
<td>2 seconds</td>
</tr>
<tr>
<td>nitrogen-13</td>
<td>10 minutes</td>
</tr>
<tr>
<td>radon-222</td>
<td>92 hours</td>
</tr>
<tr>
<td>polonium-210</td>
<td>138 days</td>
</tr>
<tr>
<td>cesium-137</td>
<td>30 years</td>
</tr>
<tr>
<td>carbon-14</td>
<td>5730 years</td>
</tr>
<tr>
<td>plutonium-239</td>
<td>24,110 years</td>
</tr>
</tbody>
</table>

*Half-life of some radioactive isotopes.*

Some of the isotopes in this table are human creations that do not exist in nature. For example, the nitrogen on Earth is almost entirely nitrogen-14 (7 protons and 7 neutrons), which is not radioactive and does not decay. The nitrogen-13 listed here has 7 protons and 6 neutrons; it can be created in a laboratory, but it is radioactive and half of it will decay within 10 minutes.

If a radioactive isotope has a half-life of \( h \) time units (which might be seconds, minutes, hours, days, or years), then after \( t \) time units the number of atoms of this isotope is reduced by half \( t/h \) times. Thus after \( t \) time units, the remaining number of atoms of the isotope will equal the original number of atoms divided by \( 2^{t/h} \). Because \( \frac{1}{2^{t/h}} = 2^{-t/h} \), we have the following result:

**Radioactive decay**

If a radioactive isotope has half-life \( h \), then the function modeling the number of atoms in a sample of this isotope is

\[
a(t) = a_0 \cdot 2^{-t/h},
\]

where \( a_0 \) is the number of atoms of the isotope in the sample at time 0.

The radioactive decay of carbon-14 has led to a clever way of determining the age of fossils, wood, and other remnants of plants and animals. Carbon-12, by far the most common form of carbon on Earth, is not radioactive and does not decay. Radioactive carbon-14 is produced regularly as cosmic rays hit the upper atmosphere. Radioactive carbon-14 then drifts down, getting into plants through photosynthesis and then into animals that eat plants and then into animals that eat animals that eat plants, and so on. Carbon-14 accounts for about \( 10^{-10} \) percent of the carbon atoms in living plants and animals.

When a plant or animal dies, it stops absorbing new carbon because it is no longer engaging in photosynthesis or eating. Thus no new carbon-14 is absorbed. The radioactive carbon-14 in the plant or animal decays, with half of it gone after 5730 years, as shown in the table above. By measuring the amount of carbon-14 as a percentage of the total amount of carbon in the remains of a plant or animal, we can then determine how long ago it died.

The 1960 Nobel Prize in Chemistry was awarded to Willard Libby for his invention of this carbon-14 dating method.
EXAMPLE 2

Suppose a cat skeleton found in an old well has a ratio of carbon-14 to carbon-12 that is 61% of the corresponding ratio for living organisms. Approximately how long ago did the cat die?

SOLUTION If $t$ denotes the number of years since the cat died, then

$$0.61 = 2^{-t/5730}.$$  

To solve this equation for $t$, take the logarithm of both sides, getting

$$\log 0.61 = -\frac{t}{5730} \log 2.$$  

Solve this equation for $t$, getting

$$t = -5730 \frac{\log 0.61}{\log 2} \approx 4086.$$  

Because we started with only two-digit accuracy (61%), we should not produce such an exact-looking estimate. Thus we might estimate that the skeleton is about 4100 years old.

Change of Base

If you want to use a calculator to evaluate something like $\log_2 73.9$, then you probably need a formula for converting logarithms from one base to another. To derive this formula, suppose $a$, $b$, and $y$ are positive numbers, with $a \neq 1$ and $b \neq 1$. Let

$$x = \log_b y.$$  

Then the definition of logarithm base $b$ implies

$$b^x = y.$$  

Take the logarithm base $a$ of both sides of the equation above, getting

$$x \log_a b = \log_a y.$$  

Thus

$$x = \frac{\log_a y}{\log_a b}.$$  

Replacing $x$ in the equation above with its value $\log_b y$ gives the following formula for converting logarithms from one base to another:

Change of base for logarithms

If $a$, $b$, and $y$ are positive numbers, with $a \neq 1$ and $b \neq 1$, then

$$\log_b y = \frac{\log_a y}{\log_a b}.$$
A special case of this formula, suitable for use with calculators, is to take 
\( a = 10 \), thus using common logarithms and getting the following formula:

**Change of base with common logarithms**

If \( b \) and \( y \) are positive numbers, with \( b \neq 1 \), then

\[
\log_b y = \frac{\log y}{\log b}.
\]

Evaluate \( \log_2 73.9 \).

**SOLUTION** Use a calculator with \( b = 2 \) and \( y = 73.9 \) in the formula above, getting

\[
\log_2 73.9 = \frac{\log 73.9}{\log 2} \approx 6.2075.
\]

The change-of-base formula for logarithms implies that the graph of the logarithm using any base can be obtained by vertically stretching the graph of the logarithm using any other base (assuming both bases are bigger than 1), as shown in the following example.

Sketch the graphs of \( \log_2 x \) and \( \log x \) on the interval \([\frac{1}{8}, 8]\). What is the relationship between these two graphs?

**SOLUTION** The change-of-base formula implies that \( \log_2 x = (\log x)/(\log 2) \). Because \( 1/(\log 2) \approx 3.32 \), this means that the graph of \( \log_2 x \) is obtained from the graph of \( \log x \) by stretching vertically by a factor of approximately 3.32.

More generally, for fixed positive numbers \( a \) and \( b \), neither of which is 1, the change-of-base formula

\[
\log_a x = \left(\log_a \ b\right) \log_b x
\]

implies that the graph of \( \log_a x \) can be obtained from the graph of \( \log_b x \) by stretching vertically by a factor of \( \log_a b \).
EXERCISES

The next two exercises emphasize that \( \log(x^y) \) does not equal \( (\log x)^y \).

1. For \( x = 5 \) and \( y = 2 \), evaluate each of the following:
   (a) \( \log(x^y) \)
   (b) \( (\log x)^y \)

2. For \( x = 2 \) and \( y = 3 \), evaluate each of the following:
   (a) \( \log(x^y) \)
   (b) \( (\log x)^y \)

3. Suppose \( y \) is such that \( \log_2 y = 17.67 \). Evaluate \( \log_2(y^{100}) \).

4. Suppose \( x \) is such that \( \log_6 x = 23.41 \). Evaluate \( \log_6(x^{10}) \).

For Exercises 5–8, find all numbers \( x \) such that the indicated equation holds.

5. \( 3^x = 8 \)
6. \( 7^x = 5 \)
7. \( 6^{\sqrt{x}} = 2 \)
8. \( 5^{\sqrt{x}} = 9 \)

9. Suppose \( m \) is a positive integer such that \( \log m \approx 13.2 \). How many digits does \( m^3 \) have?

10. Suppose \( M \) is a positive integer such that \( \log M \approx 50.3 \). How many digits does \( M^4 \) have?

11. How many digits does \( 7^{4000} \) have?

12. How many digits does \( 8^{4444} \) have?

13. Find an integer \( k \) such that \( 18^k \) has 357 digits.

14. Find an integer \( n \) such that \( 22^n \) has 222 digits.

15. Find an integer \( m \) such that \( m^{1234} \) has 1991 digits.

16. Find an integer \( N \) such that \( N^{4321} \) has 6041 digits.

17. Find the smallest integer \( n \) such that \( 7^n > 10^{100} \).

18. Find the smallest integer \( k \) such that \( 9^k > 10^{1000} \).

19. Find the smallest integer \( M \) such that \( 5^{1/M} < 1.01 \).

20. Find the smallest integer \( m \) such that \( 8^{1/m} < 1.001 \).

21. Suppose \( \log_8(\log_7 m) = 5 \). How many digits does \( m \) have?

22. Suppose \( \log_5(\log_9 m) = 6 \). How many digits does \( m \) have?

A prime number is an integer greater than 1 that has no divisors other than itself and 1.

23. At the time this book was written, the third largest known prime number was \( 2^{37156667} - 1 \). How many digits does this prime number have?

24. At the time this book was written, the second largest known prime number was \( 2^{42643801} - 1 \). How many digits does this prime number have?

25. About how many hours will it take for a sample of radon-222 to have only one-eighth as much radon-222 as the original sample?

26. About how many minutes will it take for a sample of nitrogen-13 to have only one sixty-fourth as much nitrogen-13 as the original sample?

27. About how many years will it take for a sample of cesium-137 to have only two-thirds as much cesium-137 as the original sample?

28. About how many years will it take for a sample of plutonium-239 to have only 1% as much plutonium-239 as the original sample?

29. Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope.

30. Suppose a radioactive isotope is such that five-sixths of the atoms in a sample decay after four days. Find the half-life of this isotope.

31. Suppose the ratio of carbon-14 to carbon-12 in a mummified cat is 64% of the corresponding ratio for living organisms. About how long ago did the cat die?

32. Suppose the ratio of carbon-14 to carbon-12 in a fossilized wooden tool is 20% of the corresponding ratio for living organisms. About how old is the wooden tool?

For Exercises 33–40, evaluate the indicated quantities. Your calculator probably cannot evaluate logarithms using any of the bases in these exercises, so you will need to use an appropriate change-of-base formula.

33. \( \log_2 13 \)
34. \( \log_4 27 \)
35. \( \log_{13} 9.72 \)
36. \( \log_{17} 12.31 \)
37. \( \log_9 0.23 \)
38. \( \log_7 0.58 \)
39. \( \log_{43.8} 7.1 \)
40. \( \log_{5.06} 99.2 \)
PROBLEMS

41 Explain why there does not exist an integer \( m \) such that \( 67^m \) has 9236 digits.

42 Do a web search to find the largest currently known prime number. Then calculate the number of digits in this number.

43 Suppose \( f(x) = \log x \) and \( g(x) = \log(x^4) \), with the domain of both \( f \) and \( g \) being the set of positive numbers. Explain why the graph of \( g \) can be obtained by vertically stretching the graph of \( f \) by a factor of 4.

44 Explain why

\[
\log_b \sqrt[3]{27} = \frac{\log 3}{3} = \frac{\sqrt[3]{27}}{3}.
\]

for every positive number \( b \neq 1 \).

45 Suppose \( x \) and \( b \) are positive numbers with \( b \neq 1 \). Show that if \( x \neq \sqrt[3]{27} \), then

\[
\log_{\frac{b}{3}} x \neq \log_{\frac{x}{3}} x.
\]

46 Find a positive number \( x \) such that

\[
\log_{\frac{b}{4}} x = \log_{\frac{x}{4}} x
\]

for every positive number \( b \neq 1 \).

47 Explain why expressing a large positive integer in binary notation (base 2) should take approximately 3.3 times as many digits as expressing the same positive integer in standard decimal notation (base 10).

\[
\text{For example, this problem predicts that five trillion, which requires the } 13 \text{ digits } 5,000,000,000,000 \text{ to express in decimal notation, should require approximately } 13 \times 3.3 \text{ digits (which equals 42.9 digits) to express in binary notation. Expressing five trillion in binary notation actually requires 43 digits.}
\]

48 Suppose \( a \) and \( b \) are positive numbers, with \( a \neq 1 \) and \( b \neq 1 \). Show that

\[
\log_a b = \frac{1}{\log_b a}.
\]

WORKED-OUT SOLUTIONS to Odd-Numbered Exercises

The next two exercises emphasize that \( \log(x^y) \) does not equal \( (\log x)^y \).

1 For \( x = 5 \) and \( y = 2 \), evaluate each of the following:

(a) \( \log(x^y) \)

(b) \( (\log x)^y \)

SOLUTION

(a) \( \log(5^2) = \log 25 \approx 1.39794 \)

(b) \( (\log 5)^2 \approx (0.69897)^2 \approx 0.48856 \)

3 Suppose \( y \) is such that \( \log_2 y = 17.67 \). Evaluate \( \log_2(y^{100}) \).

SOLUTION

\[
\log_2(y^{100}) = 100 \log_2 y
\]

\[
= 100 \cdot 17.67
\]

\[
= 1767
\]

For Exercises 5–8, find all numbers \( x \) such that the indicated equation holds.

5 \( 3^x = 8 \)

SOLUTION Take the common logarithm of both sides, getting \( \log(3^x) = \log 8 \), which can be rewritten as \( x \log 3 = \log 8 \). Thus

\[
x = \frac{\log 8}{\log 3} \approx 1.89279.
\]

7 \( 6^{\sqrt{x}} = 2 \)

SOLUTION Take the common logarithm of both sides, getting \( \log(6^{\sqrt{x}}) = \log 2 \), which can be rewritten as \( \sqrt{x} \log 6 = \log 2 \). Thus

\[
\sqrt{x} = \frac{\log 2}{\log 6}
\]

Hence

\[
x = \left( \frac{\log 2}{\log 6} \right)^2 \approx 0.149655.
\]

9 Suppose \( m \) is a positive integer such that \( \log m \approx 13.2 \). How many digits does \( m^3 \) have?

SOLUTION Note that
11 How many digits does \(7^{4000}\) have?

**SOLUTION** Using the formula for the logarithm of a power and a calculator, we have

\[
\log(7^{4000}) = 4000 \log 7 \approx 3380.39.
\]

Thus \(7^{4000}\) has 3381 digits.

13 Find an integer \(k\) such that \(18^k\) has 357 digits.

**SOLUTION** We want to find an integer \(k\) such that

\[
356 \leq \log(18^k) < 357.
\]

Using the formula for the logarithm of a power, we can rewrite the inequalities above as

\[
356 \leq k \log 18 < 357.
\]

Dividing by \(\log 18\) gives

\[
\frac{356}{\log 18} \leq k < \frac{357}{\log 18}.
\]

Using a calculator, we see that \(\frac{356}{\log 18} \approx 283.6\) and \(\frac{357}{\log 18} \approx 284.4\). Thus the only possible choice is to take \(k = 284\).

Again using a calculator, we see that

\[
\log(18^{284}) = 284 \log 18 \approx 356.5.
\]

Thus \(18^{284}\) indeed has 357 digits.

15 Find an integer \(m\) such that \(m^{1234}\) has 1991 digits.

**SOLUTION** We want to find an integer \(m\) such that

\[
1990 \leq \log(m^{1234}) < 1991.
\]

Using the formula for the logarithm of a power, we can rewrite the inequalities above as

\[
1990 \leq 1234 \log m < 1991.
\]

Dividing by 1234 gives

\[
\frac{1990}{1234} \leq \log m < \frac{1991}{1234}.
\]

Thus

\[
10^{\frac{1990}{1234}} \leq m < 10^{\frac{1991}{1234}}.
\]

Using a calculator, we see that \(10^{\frac{1990}{1234}} \approx 40.99\) and \(10^{\frac{1991}{1234}} \approx 41.06\). Thus the only possible choice is to take \(m = 41\).

Again using a calculator, we see that

\[
\log(41^{1234}) = 1234 \log 41 \approx 1990.18.
\]

Thus \(41^{1234}\) indeed has 1991 digits.

17 Find the smallest integer \(n\) such that \(7^n > 10^{100}\).

**SOLUTION** Suppose \(7^n > 10^{100}\). Taking the common logarithm of both sides, we have

\[
\log(7^n) > \log(10^{100}),
\]

which can be rewritten as

\[
n \log 7 > 100.
\]

This implies that

\[
n > \frac{100}{\log 7} \approx 118.33.
\]

The smallest integer that is bigger than 118.33 is 119. Thus we take \(n = 119\).
At the time this book was written, the third largest known prime number was $2^{37156667} - 1$. How many digits does this prime number have?

**SOLUTION** To calculate the number of digits in $2^{37156667} - 1$, we need to evaluate $\log(2^{37156667} - 1)$. However, $2^{37156667} - 1$ is too large to evaluate directly on a calculator, and no formula exists for the logarithm of the difference of two numbers.

The trick here is to note that $2^{37156667}$ and $2^{37156667} - 1$ have the same number of digits, as we will now see. Although it is possible for a number and the number minus 1 to have a different number of digits (for example, 100 and 99 do not have the same number of digits), this happens only if the larger of the two numbers consists of 1 followed by a bunch of 0’s and the smaller of the two numbers consists of all 9’s. Here are three different ways to see that this situation does not apply to $2^{37156667}$ and $2^{37156667} - 1$ (pick whichever explanation seems easiest to you): (a) $2^{37156667}$ cannot end in a 0 because all positive integer powers of 2 end in either 2, 4, 6, or 8; (b) $2^{37156667}$ cannot end in a 0 because then it would be divisible by 5, but $2^{37156667}$ is divisible only by integer powers of 2; (c) $2^{37156667} - 1$ cannot consist of all 9’s because then it would be divisible by 9, which is not possible for a prime number.

Now that we know that $2^{37156667}$ and $2^{37156667} - 1$ have the same number of digits, we can calculate the number of digits by taking the logarithm of $2^{37156667}$ and using the formula for the logarithm of a power. We have

$$\log(2^{37156667}) = 37156667 \log 2 \approx 11185271.3.$$  

Thus $2^{37156667}$ has 11,185,272 digits; hence $2^{37156667} - 1$ also has 11,185,272 digits.

About how many hours will it take for a sample of radon-222 to have only one-eighth as much radon-222 as the original sample?

**SOLUTION** The half-life of radon-222 is about 92 hours, as shown in the chart in this section. To reduce the number of radon-222 atoms to one-eighth the original number, we need 3 half-lives (because $2^3 = 8$). Thus it will take 276 hours (because $92 \times 3 = 276$) to have only one-eighth as much radon-222 as the original sample.

About how many years will it take for a sample of cesium-137 to have only two-thirds as much cesium-137 as the original sample?

**SOLUTION** The half-life of cesium-137 is about 30 years, as shown in the chart in this section. Thus if we start with $a$ atoms of cesium-137 at time 0, then after $t$ years there will be

$$a \cdot 2^{-t/30}$$

atoms left. We want this to equal $\frac{2}{3} a$. Thus we must solve the equation

$$a \cdot 2^{-t/30} = \frac{2}{3} a.$$  

To solve this equation for $t$, divide both sides by $a$ and then take the logarithm of both sides, getting

$$-\frac{t}{30} \log 2 = \log \frac{2}{3}.$$  

Now multiply both sides by $-1$, replace $-\log \frac{2}{3}$ by $\log \frac{3}{2}$, and then solve for $t$, getting

$$t = 30 \log \frac{3}{2} \approx 17.5.$$  

Thus two-thirds of the original sample will be left after approximately 17.5 years.

Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope.

**SOLUTION** Let $h$ denote the half-life of this isotope, measured in years. If we start with a sample of $a$ atoms of this isotope, then after 3 years there will be

$$a \cdot 2^{-3/h}$$

atoms left. We want this to equal $\frac{4}{5} a$. Thus we must solve the equation

$$a \cdot 2^{-3/h} = \frac{4}{5} a.$$  

To solve this equation for $h$, divide both sides by $a$ and then take the logarithm of both sides, getting

$$-\frac{3}{h} \log 2 = \log \frac{4}{5}.$$  

Now multiply both sides by $-1$, replace $-\log \frac{4}{5}$ by $\log \frac{5}{4}$, and then solve for $h$, getting

$$h = \frac{3 \log \frac{5}{4}}{\log 2} \approx 9.3.$$  

Thus the half-life of this isotope is approximately 9.3 years.

Suppose the ratio of carbon-14 to carbon-12 in a mummified cat is 64% of the corresponding ratio for living organisms. About how long ago did the cat die?

**SOLUTION** The half-life of carbon-14 is 5730 years. If we start with a sample of $a$ atoms of carbon-14, then after $t$ years there will be

$$a \cdot 2^{-t/5730}$$
atoms left. We want to find $t$ such that this equals $0.64a$. Thus we must solve the equation

$$a \cdot 2^{-t/5730} = 0.64a.$$

To solve this equation for $t$, divide both sides by $a$ and then take the logarithm of both sides, getting

$$-\frac{t}{5730} \log 2 = \log 0.64.$$

Now solve for $t$, getting

$$t = -5730 \frac{\log 0.64}{\log 2} \approx 3689.$$

Thus the cat died about 3689 years ago. Carbon-14 cannot be measured with extreme accuracy. Thus it is better to estimate that the cat died about 3700 years ago (because a number such as 3689 conveys more accuracy than will be present in such measurements).

For Exercises 33–40, evaluate the indicated quantities. Your calculator probably cannot evaluate logarithms using any of the bases in these exercises, so you will need to use an appropriate change-of-base formula.

33 \( \log_2 13 \)

**SOLUTION** \( \log_2 13 = \frac{\log 13}{\log 2} \approx 3.70044 \)

35 \( \log_{13} 9.72 \)

**SOLUTION** \( \log_{13} 9.72 = \frac{\log 9.72}{\log 13} \approx 0.88664 \)

37 \( \log_9 0.23 \)

**SOLUTION** \( \log_9 0.23 = \frac{\log 0.23}{\log 9} \approx -0.668878 \)

39 \( \log_{4.38} 7.1 \)

**SOLUTION** \( \log_{4.38} 7.1 = \frac{\log 7.1}{\log 4.38} \approx 1.32703 \)

The author’s second cat proofreading the manuscript.
Applications of the Product and Quotient Rules for Logarithms

LEARNING OBJECTIVES
By the end of this section you should be able to

- apply the formula for the logarithm of a product;
- apply the formula for the logarithm of a quotient;
- model earthquake intensity with the logarithmic Richter magnitude scale;
- model sound intensity with the logarithmic decibel scale;
- model star brightness with the logarithmic apparent magnitude scale.

Logarithm of a Product

Logarithms convert products to sums. To see this, suppose \( b, x, \) and \( y \) are positive numbers, with \( b \neq 1 \). Let

\[
 u = \log_b x \quad \text{and} \quad v = \log_b y.
\]

Then the definition of logarithm base \( b \) implies

\[
 b^u = x \quad \text{and} \quad b^v = y.
\]

Multiplying together these equations and using the identity \( b^{u+v} = b^u b^v \) gives

\[
 b^{u+v} = xy.
\]

Again using the definition of logarithm base \( b \), the equation above implies

\[
 \log_b (xy) = u + v
\]

\[
 = \log_b x + \log_b y.
\]

Thus we have the following formula for the logarithm of a product:

**Logarithm of a product**

If \( b, x, \) and \( y \) are positive numbers, with \( b \neq 1 \), then

\[
 \log_b (xy) = \log_b x + \log_b y.
\]

Use the information that \( \log 6 \approx 0.778 \) to evaluate \( \log 60000 \).

**SOLUTION**

\[
 \log 60000 = \log (10^4 \cdot 6) \\
= \log (10^4) + \log 6 \\
= 4 + \log 6 \\
\approx 4.778.
\]
Logarithm of a Quotient

Logarithms convert quotients to differences. To see this, suppose \( b, x, \) and \( y \) are positive numbers, with \( b \neq 1 \). Let

\[
    u = \log_b x \quad \text{and} \quad v = \log_b y.
\]

Then the definition of logarithm base \( b \) implies

\[
    b^u = x \quad \text{and} \quad b^v = y.
\]

Divide the first equation by the second equation and use the identity \( \frac{b^u}{b^v} = b^{u-v} \) to get

\[
    b^{u-v} = \frac{x}{y}.
\]

Again using the definition of logarithm base \( b \), the equation above implies

\[
    \log_b \frac{x}{y} = u - v = \log_b x - \log_b y.
\]

Thus we have the following formula for the logarithm of a quotient:

\[
    \log_b \frac{x}{y} = \log_b x - \log_b y.
\]

Example 2

Suppose \( \log_4 x = 8.9 \) and \( \log_4 y = 2.2 \). Evaluate \( \log_4 \frac{4x}{y} \).

Solution

Using the quotient and product rules, we have

\[
    \log_4 \frac{4x}{y} = \log_4 (4x) - \log_4 y
\]

\[
    = \log_4 4 + \log_4 x - \log_4 y
\]

\[
    = 1 + 8.9 - 2.2
\]

\[
    = 7.7.
\]

As a special case of the formula for the logarithm of a quotient, take \( x = 1 \) in the formula above for the logarithm of a quotient, getting

\[
    \log_b \frac{1}{y} = \log_b 1 - \log_b y.
\]

Recalling that \( \log_b 1 = 0 \), we get the following result:

Logarithm of a multiplicative inverse

\[
    \log_b \frac{1}{y} = -\log_b y.
\]
Earthquakes and the Richter Scale

The intensity of an earthquake is measured by the size of the seismic waves generated by the earthquake. These numbers vary across such a huge scale that earthquakes are usually reported using the Richter magnitude scale, which is a logarithmic scale using common logarithms (base 10).

**Richter magnitude scale**

An earthquake with seismic waves of size \( S \) has **Richter magnitude**

\[
\log \frac{S}{S_0},
\]

where \( S_0 \) is the size of the seismic waves corresponding to what has been declared to be an earthquake with Richter magnitude 0.

A few points will help clarify this definition:

- The value of \( S_0 \) was set in 1935 by the American seismologist Charles Richter as approximately the size of the smallest seismic waves that could be measured at that time.
- The unit used to measure \( S \) and \( S_0 \) does not matter because any change in the scale of this unit disappears in the ratio \( \frac{S}{S_0} \).
- An increase of earthquake intensity by a factor of 10 corresponds to an increase of 1 in Richter magnitude, as can be seen from the equation

\[
\log \frac{10S}{S_0} = \log 10 + \log \frac{S}{S_0} = 1 + \log \frac{S}{S_0}.
\]

The world’s most intense recorded earthquake struck Chile in 1960 with Richter magnitude 9.5. The most intense recorded earthquake in the United States struck Alaska in 1964 with Richter magnitude 9.2. Approximately how many times more intense was the 1960 earthquake in Chile than the 1964 earthquake in Alaska?

**Solution**  Let \( S_C \) denote the size of the seismic waves from the 1960 earthquake in Chile and let \( S_A \) denote the size of the seismic waves from the 1964 earthquake in Alaska. Thus

\[
9.5 = \log \frac{S_C}{S_0} \quad \text{and} \quad 9.2 = \log \frac{S_A}{S_0}.
\]

Subtracting the second equation from the first equation, we get

\[
0.3 = \log \frac{S_C}{S_0} - \log \frac{S_A}{S_0} = \log \left( \frac{S_C}{S_0} / \frac{S_A}{S_0} \right) = \log \frac{S_C}{S_A}.
\]

Thus

\[
\frac{S_C}{S_A} = 10^{0.3} \approx 2.
\]

In other words, the 1960 earthquake in Chile was approximately twice as intense as the 1964 earthquake in Alaska.
Sound Intensity and Decibels

The ratio of the intensity of sound causing pain to the intensity of the quietest sound we can hear is over one trillion. Working with such large numbers can be inconvenient. Thus sound is measured in a logarithmic scale called decibels.

**Decibel scale for sound**

A sound with intensity \( E \) has

\[
10 \log \frac{E}{E_0}
\]

decibels, where \( E_0 \) is the intensity of an extremely quiet sound at the threshold of human hearing.

A few points will help clarify this definition:

- The value of \( E_0 \) is \( 10^{-12} \) watts per square meter.
- The intensity of sound is usually measured in watts per square meter, but the unit used to measure \( E \) and \( E_0 \) does not matter because any change in the scale of this unit disappears in the ratio \( \frac{E}{E_0} \).
- Multiplying sound intensity by a factor of 10 corresponds to adding 10 to the decibel measurement, as can be seen from the equation

\[
10 \log 10 \frac{10E}{E_0} = 10 \log 10 + 10 \log \frac{E}{E_0} = 10 + 10 \log \frac{E}{E_0}.
\]

**Example 4**

French law limits iPods and MP3 players to a maximum possible volume of 100 decibels. Normal conversation has a sound level of 65 decibels. How many more times intense than normal conversation is a sound of 100 decibels?

**Solution**  Let \( E_F \) denote the sound intensity of 100 decibels allowed in France and let \( E_C \) denote the sound intensity of normal conversation. Thus

\[
100 = 10 \log \frac{E_F}{E_0} \quad \text{and} \quad 65 = 10 \log \frac{E_C}{E_0}.
\]

Subtracting the second equation from the first equation, we get

\[
35 = 10 \log \frac{E_F}{E_0} - 10 \log \frac{E_C}{E_0}.
\]

Thus

\[
3.5 = \log \frac{E_F}{E_0} - \log \frac{E_C}{E_0} = \log \left( \frac{E_F}{E_0} / \frac{E_C}{E_0} \right) = \log \frac{E_F}{E_C}.
\]

Thus

\[
\frac{E_F}{E_C} = 10^{3.5} \approx 3162.
\]

Hence an iPod operating at the maximum legal French volume of 100 decibels produces sound about three thousand times more intense than normal conversation.
The increase in sound intensity by a factor of more than 3000 in the last example is not as drastic as it seems because of how we perceive loudness:

**Loudness**

The human ear perceives each increase in sound by 10 decibels to be a doubling of loudness (even though the sound intensity has actually increased by a factor of 10).

By what factor has the loudness increased in going from normal speech at 65 decibels to an iPod at 100 decibels?

**SOLUTION** Here we have an increase of 35 decibels, so we have had an increase of 10 decibels 3.5 times. Thus the perceived loudness has doubled 3.5 times, which means that it has increased by a factor of $2^{3.5}$. Because $2^{3.5} \approx 11$, this means that an iPod operating at 100 decibels seems about 11 times louder than normal conversation.

**Star Brightness and Apparent Magnitude**

The ancient Greeks divided the visible stars into six groups based on their brightness. The brightest stars were called first-magnitude stars. The next brightest group of stars were called second-magnitude stars, and so on, until the sixth-magnitude stars consisted of the barely visible stars.

About two thousand years later, astronomers made the ancient Greek star magnitude scale more precise. The typical first-magnitude stars were about 100 times brighter than the typical sixth-magnitude stars. Because there are five steps in going from the first magnitude to the sixth magnitude, this means that with each magnitude the brightness should decrease by a factor of $100^{1/5}$.

Originally the scale was defined so that Polaris (the North Star) had magnitude 2. If we let $b_2$ denote the brightness of Polaris, this would mean that a third-magnitude star has brightness $b_2/100^{1/5}$, a fourth-magnitude star has brightness $b_2/(100^{1/5})^2$, a fifth-magnitude star has brightness $b_2/(100^{1/5})^3$, and so on. Thus the brightness $b$ of a star with magnitude $m$ should be given by the equation

$$b = \frac{b_2}{(100^{1/5})^{(m-2)}} = b_2 100^{(2-m)/5} = b_2 100^{2/5} 100^{-m/5} = b_0 100^{-m/5},$$

where $b_0 = b_2 100^{2/5}$. If we divide both sides of the equation above by $b_0$ and then take logarithms we get

$$\log \frac{b}{b_0} = \log(100^{-m/5}) = -\frac{m}{5} \log 100 = -\frac{2m}{5}.$$

Solving this equation for $m$ leads to the following definition:
**Apparent magnitude**

An object with brightness $b$ has **apparent magnitude**

$$\frac{5}{2} \log \frac{b_0}{b},$$

where $b_0$ is the brightness of an object with magnitude 0.

A few points will help clarify this definition:

- The term “apparent magnitude” is more accurate than “magnitude” because we are measuring how bright a star appears from Earth. A glowing luminous star might appear dim from Earth because it is very far away.
- Although this apparent magnitude scale was originally set up for stars, it can be applied to other objects such as the full moon.
- Although the value of $b_0$ was originally set so that Polaris (the North Star) would have apparent magnitude 2, the definition has changed slightly. With the current definition of $b_0$, Polaris has magnitude close to 2 but not exactly equal to 2.
- The unit used to measure brightness does not matter because any change in the scale of this unit disappears in the ratio $\frac{b_0}{b}$.

**EXAMPLE 6**  

Because of the lack of atmospheric interference, the Hubble telescope can see dimmer stars than Earth-based telescopes of the same size.

With good binoculars you can see stars with apparent magnitude 9. The Hubble telescope, which is in orbit around the Earth, can detect stars with apparent magnitude 30. How much better is the Hubble telescope than good binoculars, measured in terms of the ratio of the brightness of stars that they can detect?

**SOLUTION**  Let $b_9$ denote the brightness of a star with apparent magnitude 9 and let $b_{30}$ denote the brightness of a star with apparent magnitude 30. Thus

$$9 = \frac{5}{2} \log \frac{b_0}{b_9} \quad \text{and} \quad 30 = \frac{5}{2} \log \frac{b_0}{b_{30}}.$$

Subtracting the first equation from the second equation, we get

$$21 = \frac{5}{2} \log \frac{b_0}{b_{30}} - \frac{5}{2} \log \frac{b_0}{b_9}.$$

Multiplying both sides of the equation above by $\frac{2}{5}$ gives

$$\frac{42}{5} = \log \frac{b_0}{b_{30}} - \log \frac{b_0}{b_9} = \log \left( \frac{b_0}{b_{30}} / \frac{b_0}{b_9} \right) = \log \frac{b_9}{b_{30}}.$$

Thus

$$\frac{b_9}{b_{30}} = 10^{42/5} = 10^{8.4} \approx 250,000,000.$$

Thus the Hubble telescope can detect stars 250 million times dimmer than stars we can see with good binoculars.
EXERCISES

The next two exercises emphasize that \( \log(x + y) \) does not equal \( \log x + \log y \).
1. For \( x = 7 \) and \( y = 13 \), evaluate:
   (a) \( \log(x + y) \)
   (b) \( \log x + \log y \)
2. For \( x = 0.4 \) and \( y = 3.5 \), evaluate:
   (a) \( \log(x + y) \)
   (b) \( \log x + \log y \)

The next two exercises emphasize that \( \log(x y) \) does not equal \( (\log x)(\log y) \).
3. For \( x = 3 \) and \( y = 8 \), evaluate:
   (a) \( \log(x y) \)
   (b) \( (\log x)(\log y) \)
4. For \( x = 1.1 \) and \( y = 5 \), evaluate:
   (a) \( \log(x y) \)
   (b) \( (\log x)(\log y) \)

The next two exercises emphasize that \( \log \frac{x}{y} \) does not equal \( \frac{\log x}{\log y} \).
5. For \( x = 12 \) and \( y = 2 \), evaluate:
   (a) \( \log \frac{x}{y} \)
   (b) \( \frac{\log x}{\log y} \)
6. For \( x = 18 \) and \( y = 0.3 \), evaluate:
   (a) \( \log \frac{x}{y} \)
   (b) \( \frac{\log x}{\log y} \)
7. How many digits does \( 6^{700} \cdot 23^{1000} \) have?
8. How many digits does \( 5^{999} \cdot 17^{2222} \) have?
9. Suppose \( m \) and \( n \) are positive integers such that \( \log m \approx 32.1 \) and \( \log n \approx 7.3 \). How many digits does \( mn \) have?
10. Suppose \( m \) and \( n \) are positive integers such that \( \log m \approx 41.3 \) and \( \log n \approx 12.8 \). How many digits does \( mn \) have?
11. Suppose \( \log a = 118.7 \) and \( \log b = 119.7 \). Evaluate \( \frac{b}{a} \).
12. Suppose \( \log a = 203.4 \) and \( \log b = 205.4 \). Evaluate \( \frac{b}{a} \).

For Exercises 13–26, evaluate the given quantities assuming that
\[ \log_3 x = 5.3 \quad \text{and} \quad \log_3 y = 2.1, \]
\[ \log_4 u = 3.2 \quad \text{and} \quad \log_4 v = 1.3. \]
13. \( \log_3(9xy) \)
14. \( \log_4(2uv) \)
15. \( \log_3 \frac{x}{y} \)
16. \( \log_4 \frac{u}{v} \)
17. \( \log_3 \sqrt{x} \)
18. \( \log_4 \sqrt{u} \)
19. \( \log_3 \frac{1}{\sqrt{y}} \)
20. \( \log_4 \frac{1}{\sqrt{v}} \)
21. \( \log_3(x^2y^3) \)
22. \( \log_4(u^3v^4) \)
23. \( \log_3 \frac{x^3}{y^2} \)
24. \( \log_4 \frac{u^2}{v^3} \)
25. \( \log_9(x^{10}) \)
26. \( \log_2(u^{100}) \)

For Exercises 27–34, find all numbers \( x \) that satisfy the given equation.
27. \( \log_7(x + 5) - \log_7(x - 1) = 2 \)
28. \( \log_4(x + 4) - \log_4(x - 2) = 3 \)
29. \( \log_3(x + 5) + \log_3(x - 1) = 2 \)
30. \( \log_5(x + 4) + \log_5(x + 2) = 2 \)
31. \( \frac{\log_6(15x)}{\log_6(5x)} = 2 \)
32. \( \frac{\log_9(13x)}{\log_9(4x)} = 2 \)
33. \( (\log(3x)) \log x = 4 \)
34. \( (\log(6x)) \log x = 5 \)
35. How many more times intense is an earthquake with Richter magnitude 7 than an earthquake with Richter magnitude 5?
36. How many more times intense is an earthquake with Richter magnitude 6 than an earthquake with Richter magnitude 3?
37. The 1994 Northridge earthquake in Southern California, which killed several dozen people, had Richter magnitude 6.7. What would be the Richter magnitude of an earthquake that was 100 times more intense than the Northridge earthquake?
38. The 1995 earthquake in Kobe (Japan), which killed over 6000 people, had Richter magnitude 7.2. What would be the Richter magnitude of an earthquake that was 1000 times less intense than the Kobe earthquake?
39. The most intense recorded earthquake in New York state was in 1944; it had Richter magnitude 5.8. The most intense recorded earthquake in Minnesota was in 1975; it had Richter magnitude 5.0. How many times more intense was the 1944 earthquake in New York than the 1975 earthquake in Minnesota?
40. The most intense recorded earthquake in Wyoming was in 1959; it had Richter magnitude 6.5. The most intense recorded earthquake in Illinois was in 1968; it had Richter magnitude 5.3. How many times more intense was the 1959 earthquake in Wyoming than the 1968 earthquake in Illinois?
41. The most intense recorded earthquake in Texas occurred in 1931; it had Richter magnitude 5.8. If an earthquake were to strike Texas next year that was three times more intense than the current record in Texas, what would its Richter magnitude be?
42 The most intense recorded earthquake in Ohio occurred in 1937; it had Richter magnitude 5.4. If an earthquake were to strike Ohio next year that was 1.6 times more intense than the current record in Ohio, what would its Richter magnitude be?

43 Suppose you whisper at 20 decibels and normally speak at 60 decibels.
   (a) Find the ratio of the sound intensity of your normal speech to the sound intensity of your whisper.
   (b) How many times louder does your normal speech seem than your whisper?

44 Suppose your vacuum cleaner produces a sound of 80 decibels and you normally speak at 60 decibels.
   (a) Find the ratio of the sound intensity of your vacuum cleaner to the sound intensity of your normal speech.
   (b) How many times louder does your vacuum cleaner seem than your normal speech?

45 Suppose an airplane taking off makes a noise of 117 decibels and you normally speak at 63 decibels.
   (a) Find the ratio of the sound intensity of the airplane to the sound intensity of your normal speech.
   (b) How many times louder does the airplane seem than your normal speech?

46 Suppose your cell phone rings at a noise level of 74 decibels and you normally speak at 61 decibels.
   (a) Find the ratio of the sound intensity of your cell phone ring to the sound intensity of your normal speech.
   (b) How many times louder does your cell phone ring seem than your normal speech?

47 Suppose a television is playing softly at a sound level of 50 decibels. What decibel level would make the television sound eight times as loud?

48 Suppose a radio is playing loudly at a sound level of 80 decibels. What decibel level would make the radio sound one-fourth as loud?

49 Suppose a motorcycle produces a sound level of 90 decibels. What decibel level would make the motorcycle sound one-third as loud?

50 Suppose a rock band is playing loudly at a sound level of 100 decibels. What decibel level would make the band sound three-fifths as loud?

51 How many times brighter is a star with apparent magnitude 2 than a star with apparent magnitude 17?

52 How many times brighter is a star with apparent magnitude 3 than a star with apparent magnitude 23?

53 Sirius, the brightest star that can be seen from Earth (not counting the sun), has an apparent magnitude of \(-1.4\). Vega, which was the North Star about 12,000 years ago (slight changes in Earth’s orbit lead to changing North Stars every several thousand years), has an apparent magnitude of 0.03. How many times brighter than Vega is Sirius?

54 The full moon has an apparent magnitude of approximately \(-12.6\). How many times brighter than Sirius is the full moon?

55 Neptune has an apparent magnitude of about 7.8. What is the apparent magnitude of a star that is 20 times brighter than Neptune?

56 What is the apparent magnitude of a star that is eight times brighter than Neptune?

PROBLEMS

57 Explain why \( \log 500 = 3 - \log 2 \).

58 Explain why \( \log \sqrt{0.07} = \frac{\log 7}{2} - 1 \).

59 Explain why \( 1 + \log x = \log(10x) \)
   for every positive number \( x \).

60 Explain why \( 2 - \log x = \log \frac{100}{x} \)
   for every positive number \( x \).

61 Explain why
   \begin{equation}
   (1 + \log x)^2 = \log(10x^2) + (\log x)^2
   \end{equation}
   for every positive number \( x \).

62 Explain why \( \frac{1 + \log x}{2} = \log \sqrt{10x} \)
   for every positive number \( x \).

63 Suppose \( f(x) = \log x \) and \( g(x) = \log(1000x) \). Explain why the graph of \( g \) can be obtained by shifting the graph of \( f \) up 3 units.
64 Suppose \( f(x) = \log_b x \) and \( g(x) = \log_b \frac{10}{x} \). Explain why the graph of \( g \) can be obtained by flipping the graph of \( f \) across the horizontal axis and then shifting up 2 units.

65 Suppose \( f(x) = \log x \) and \( g(x) = \log(100x^3) \). Explain why the graph of \( g \) can be obtained by vertically stretching the graph of \( f \) by a factor of 3 and then shifting up 2 units.

66 Show that an earthquake with Richter magnitude \( R \) has seismic waves of size \( S_010^R \), where \( S_0 \) is the size of the seismic waves of an earthquake with Richter magnitude 0.

67 Do a web search to find the most intense earthquake in Japan in the last calendar year and the most intense earthquake in Japan in the last calendar year. Approximately how many times more intense was the larger of these two earthquakes than the smaller of the two?

68 Show that a sound with \( d \) decibels has intensity \( E_010^{d/10} \), where \( E_0 \) is the intensity of a sound with 0 decibels.

69 Find at least three different web sites giving the apparent magnitude of Polaris (the North Star) accurate to at least two digits after the decimal point. If you find different values on different web sites (as the author did), then try to explain what could account for the discrepancy (and take this as a good lesson in the caution necessary when using the web as a source of scientific information).

70 Write a description of the logarithmic scale used for the pH scale, which measures acidity (this will probably require use of the library or the web).

71 Without doing any calculations, explain why the solutions to the equations in Exercises 31 and 32 are unchanged if we change the base for all the logarithms in those exercises to any positive number \( b \neq 1 \).

72 Explain why the equation

\[
\log \frac{x - 3}{x - 2} = 2
\]

has a solution but the equation

\[
\log(x - 3) - \log(x - 2) = 2
\]

has no solutions.

73 Pretend that you are living in the time before calculators and computers existed, and that you have a book showing the logarithms of 1.001, 1.002, 1.003, and so on, up to the logarithm of 9.999. Explain how you would find the logarithm of 457.2, which is beyond the range of your book.

74 Explain why books of logarithm tables, which were frequently used before the era of calculators and computers, gave logarithms only for numbers between 1 and 10.

75 Suppose \( b \) and \( y \) are positive numbers, with \( b \neq 1 \) and \( b \neq \frac{1}{2} \). Show that

\[
\log_b y = \frac{\log_b y}{1 + \log_b 2}.
\]

WORKED-OUT SOLUTIONS to Odd-Numbered Exercises

The next two exercises emphasize that \( \log(x + y) \) does not equal \( \log x + \log y \).

1 \( \text{For } x = 7 \text{ and } y = 13, \text{ evaluate:} \)

(a) \( \log(x + y) \)
(b) \( \log x + \log y \)

SOLUTION

(a) \( \log(7 + 13) = \log 20 \approx 1.30103 \)
(b) \( \log 7 + \log 13 \approx 0.845098 + 1.113943 = 1.959041 \)

The next two exercises emphasize that \( \log(xy) \) does not equal \( (\log x)(\log y) \).

3 \( \text{For } x = 3 \text{ and } y = 8, \text{ evaluate:} \)

(a) \( \log(xy) \)
(b) \( (\log x)(\log y) \)

SOLUTION

(a) \( \log(3 \cdot 8) = \log 24 \approx 1.38021 \)
(b) \( (\log 3)(\log 8) \approx (0.477121)(0.903090) \approx 0.430883 \)

The next two exercises emphasize that \( \log \frac{x}{y} \) does not equal \( \frac{\log x}{\log y} \).

5 \( \text{For } x = 12 \text{ and } y = 2, \text{ evaluate:} \)

(a) \( \log \frac{x}{y} \)
(b) \( \frac{\log x}{\log y} \)

SOLUTION

(a) \( \log \frac{12}{2} = \log 6 \approx 0.778151 \)
(b) \( \frac{\log 12}{\log 2} = \frac{1.079181}{0.301030} \approx 3.58496 \)
7 How many digits does \(6^{700} \cdot 23^{1000}\) have?

**SOLUTION** Using the formulas for the logarithm of a product and the logarithm of a power, we have

\[
\log(6^{700} \cdot 23^{1000}) = \log(6^{700}) + \log(23^{1000})
\]

\[
= 700 \log 6 + 1000 \log 23
\]

\[
\approx 1906.43.
\]

Thus \(6^{700} \cdot 23^{1000}\) has 1907 digits.

9 Suppose \(m\) and \(n\) are positive integers such that \(\log m \approx 32.1\) and \(\log n \approx 7.3\). How many digits does \(mn\) have?

**SOLUTION** Note that

\[
\log(mn) = \log m + \log n \approx 32.1 + 7.3 = 39.4.
\]

Thus \(mn\) has 40 digits.

11 Suppose \(\log a = 118.7\) and \(\log b = 119.7\). Evaluate \(\frac{b}{a}\).

**SOLUTION** Note that

\[
\log \left(\frac{b}{a}\right) = \log b - \log a = 119.7 - 118.7 = 1.
\]

Thus \(\frac{b}{a} = 10\).

For Exercises 13–26, evaluate the given quantities assuming that

\[
\log_3 x = 5.3 \quad \text{and} \quad \log_3 y = 2.1,
\]

\[
\log_4 u = 3.2 \quad \text{and} \quad \log_4 v = 1.3.
\]

13 \(\log_3 (9xy)\)

**SOLUTION**

\[
\log_3 (9xy) = \log_3 9 + \log_3 x + \log_3 y
\]

\[
= 2 + 5.3 + 2.1
\]

\[
= 9.4
\]

15 \(\log_3 \frac{x}{3y}\)

**SOLUTION**

\[
\log_3 \left(\frac{x}{3y}\right) = \log_3 x - \log_3 (3y)
\]

\[
= \log_3 x - \log_3 3 - \log_3 y
\]

\[
= 5.3 - 1 - 2.1
\]

\[
= 2.2
\]

17 \(\log_3 \sqrt{x}\)

**SOLUTION**

\[
\log_3 \sqrt{x} = \log_3 (x^{1/2})
\]

\[
= \frac{1}{2} \log_3 x
\]

\[
= \frac{1}{2} \times 5.3
\]

\[
= 2.65
\]

19 \(\log_3 \frac{1}{\sqrt{y}}\)

**SOLUTION**

\[
\log_3 \frac{1}{\sqrt{y}} = \log_3 (y^{-1/2})
\]

\[
= -\frac{1}{2} \log_3 y
\]

\[
= -\frac{1}{2} \times 2.1
\]

\[
= -1.05
\]

21 \(\log_3 (x^2y^3)\)

**SOLUTION**

\[
\log_3 (x^2y^3) = \log_3 (x^2) + \log_3 (y^3)
\]

\[
= 2 \log_3 x + 3 \log_3 y
\]

\[
= 2 \cdot 5.3 + 3 \cdot 2.1
\]

\[
= 16.9
\]

23 \(\log_3 \frac{x^3}{y^2}\)

**SOLUTION**

\[
\log_3 \left(\frac{x^3}{y^2}\right) = \log_3 (x^3) - \log_3 (y^2)
\]

\[
= 3 \log_3 x - 2 \log_3 y
\]

\[
= 3 \cdot 5.3 - 2 \cdot 2.1
\]

\[
= 11.7
\]

25 \(\log_9 (x^{10})\)

**SOLUTION** Because \(\log_9 x = 5.3\), we see that \(3^{5.3} = x\). This equation can be rewritten as \((9^{1/2})^{5.3} = x\), which can then be rewritten as \(9^{2.65} = x\). In other words, \(\log_9 x = 2.65\). Thus

\[
\log_9 (x^{10}) = 10 \log_9 x = 26.5
\]

For Exercises 27–34, find all numbers \(x\) that satisfy the given equation.
27. \( \log_7(x + 5) - \log_7(x - 1) = 2 \)

**SOLUTION**  
Rewrite the equation as follows:
\[
2 = \log_7(x + 5) - \log_7(x - 1)
= \log_7\left(\frac{x + 5}{x - 1}\right)
\]

Thus
\[
\frac{x + 5}{x - 1} = 7^2 = 49.
\]
We can solve the equation above for \( x \), getting \( x = \frac{9}{2} \).

29. \( \log_3(x + 5) + \log_3(x - 1) = 2 \)

**SOLUTION**  
Rewrite the equation as follows:
\[
2 = \log_3(x + 5) + \log_3(x - 1)
= \log_3((x + 5)(x - 1))
= \log_3(x^2 + 4x - 5)
\]

Thus
\[
x^2 + 4x - 5 = 3^2 = 9,
\]
which implies that
\[
x^2 + 4x - 14 = 0.
\]

We can solve the equation above using the quadratic formula, getting \( x = 3\sqrt{2} - 2 \) or \( x = -3\sqrt{2} - 2 \). However, both \( x + 5 \) and \( x - 1 \) are negative if \( x = -3\sqrt{2} - 2 \); because the logarithm of a negative number is undefined, we must discard this root of the equation above. We conclude that the only value of \( x \) satisfying the equation \( \log_3(x + 5) + \log_3(x - 1) = 2 \) is \( x = 3\sqrt{2} - 2 \).

31. \( \frac{\log_6(15x)}{\log_6(5x)} = 2 \)

**SOLUTION**  
Rewrite the equation as follows:
\[
2 = \frac{\log_6(15x)}{\log_6(5x)}
= \frac{\log_6 15 + \log_6 x}{\log_6 5 + \log_6 x}.
\]

Solving this equation for \( \log_6 x \) (the first step in doing this is to multiply both sides by the denominator \( \log_6 5 + \log_6 x \)), we get
\[
\log_6 x = \log_6 15 - 2 \log_6 5
= \log_6 15 - \log_6 25
= \log_6 \frac{15}{25}
= \log_6 \frac{3}{5}.
\]

Thus \( x = \frac{3}{5} \).

33. \( (\log(3x)) \log x = 4 \)

**SOLUTION**  
Rewrite the equation as follows:
\[
4 = (\log(3x)) \log x
= (\log x + \log 3) \log x
= (\log x)^2 + (\log 3)(\log x).
\]

Letting \( y = \log x \), we can rewrite the equation above as
\[
y^2 + (\log 3)y - 4 = 0.
\]

Use the quadratic formula to solve the equation above for \( y \), getting
\[
y \approx -2.25274 \text{ or } y \approx 1.77562.
\]

Thus
\[
\log x \approx -2.25274 \text{ or } \log x \approx 1.77562,
\]
which means that
\[
x \approx 10^{-2.25274} \approx 0.00558807
\]
or
\[
x \approx 10^{1.77562} \approx 59.6509.
\]

35. How many more times intense is an earthquake with Richter magnitude 7 than an earthquake with Richter magnitude 5?

**SOLUTION**  
Here is a more formal explanation using logarithms:  
Let \( S_7 \) denote the size of the seismic waves from an earthquake with Richter magnitude 7 and let \( S_5 \) denote the size of the seismic waves from an earthquake with Richter magnitude 5. Thus
\[
7 = \log \frac{S_7}{S_0} \text{ and } 5 = \log \frac{S_5}{S_0}.
\]

Subtracting the second equation from the first, we get
\[
2 = \log \frac{S_7}{S_0} - \log \frac{S_5}{S_0} = \log \left( \frac{S_7}{S_0} / \frac{S_5}{S_0} \right) = \log \frac{S_7}{S_5}.
\]

Thus
\[
\frac{S_7}{S_5} = 10^2 = 100.
\]

Hence an earthquake with Richter magnitude 7 is 100 times more intense than an earthquake with Richter magnitude 5.
37 The 1994 Northridge earthquake in Southern California, which killed several dozen people, had Richter magnitude 6.7. What would be the Richter magnitude of an earthquake that was 100 times more intense than the Northridge earthquake?

**SOLUTION** Each increase of 1 in the Richter magnitude corresponds to an increase in the intensity of the earthquake by a factor of 10. Hence an increase in intensity by a factor of 100 (which equals $10^2$) corresponds to an increase of 2 is the Richter magnitude. Thus an earthquake that was 100 times more intense than the Northridge earthquake would have Richter magnitude $6.7 + 2$, which equals 8.7.

39 The most intense recorded earthquake in New York state was in 1944; it had Richter magnitude 5.8. The most intense recorded earthquake in Minnesota was in 1975; it had Richter magnitude 5.0. How many times more intense was the 1944 earthquake in New York than the 1975 earthquake in Minnesota?

**SOLUTION** Let $S_N$ denote the size of the seismic waves from the 1944 earthquake in New York and let $S_M$ denote the size of the seismic waves from the 1975 earthquake in Minnesota. Thus

$$5.8 = \log \frac{S_N}{S_0} \quad \text{and} \quad 5.0 = \log \frac{S_M}{S_0}.$$ 

Subtracting the second equation from the first equation, we get

$$0.8 = \log \frac{S_N}{S_0} - \log \frac{S_M}{S_0} = \log \left( \frac{S_N}{S_0} / \frac{S_M}{S_0} \right) = \log \frac{S_N}{S_M}.$$ 

Thus

$$\frac{S_N}{S_M} = 10^{0.8} \approx 6.3.$$ 

In other words, the 1944 earthquake in New York was approximately 6.3 times more intense than the 1975 earthquake in Minnesota.

41 The most intense recorded earthquake in Texas occurred in 1931; it had Richter magnitude 5.8. If an earthquake were to strike Texas next year that was three times more intense than the current record in Texas, what would its Richter magnitude be?

**SOLUTION** Let $S_T$ denote the size of the seismic waves from the 1931 earthquake in Texas. Thus

$$5.8 = \log \frac{S_T}{S_0}.$$ 

An earthquake three times more intense would have Richter magnitude

$$\log \frac{3S_T}{S_0} = \log 3 + \log \frac{S_T}{S_0} = 0.477 + 5.8 = 6.277.$$ 

Because of the difficulty of obtaining accurate measurements, Richter magnitudes are usually reported with only one digit after the decimal place. Rounding off, we would thus say that an earthquake in Texas that was three times more intense than the current record would have Richter magnitude 6.3.

43 Suppose you whisper at 20 decibels and normally speak at 60 decibels.

(a) Find the ratio of the sound intensity of your normal speech to the sound intensity of your whisper.

(b) How many times louder does your normal speech seem than your whisper?

**SOLUTION**

(a) Each increase of 10 decibels corresponds to multiplying the sound intensity by a factor of 10. Going from a 20-decibel whisper to 60-decibel normal speech means that the sound intensity has been increased by a factor of 10 four times. Because $10^4 = 10,000$, this means that the ratio of the sound intensity of your normal speech to the sound intensity of your whisper is 10,000.

(b) Each increase of 10 decibels results in a doubling of loudness. Here we have an increase of 40 decibels, so we have had an increase of 10 decibels four times. Thus the perceived loudness has increased by a factor of $2^4$. Because $2^4 = 16$, this means that your normal conversation seems 16 times louder than your whisper.

45 Suppose an airplane taking off makes a noise of 117 decibels and you normally speak at 63 decibels.

(a) Find the ratio of the sound intensity of the airplane to the sound intensity of your normal speech.

(b) How many times louder does the airplane seem than your normal speech?

**SOLUTION**

(a) Let $E_A$ denote the sound intensity of the airplane taking off and let $E_S$ denote the sound intensity of your normal speech. Thus

$$117 = 10 \log \frac{E_A}{E_0} \quad \text{and} \quad 63 = 10 \log \frac{E_S}{E_0}.$$ 

Subtracting the second equation from the first equation, we get

$$54 = 10 \log \frac{E_A}{E_0} - 10 \log \frac{E_S}{E_0}.$$
Thus
\[
5.4 = \log \frac{E_A}{E_0} - \log \frac{E_S}{E_0} = \log \left( \frac{E_A}{E_0} / \frac{E_S}{E_0} \right) = \log \frac{E_A}{E_S}.
\]
Thus
\[
\frac{E_A}{E_S} = 10^{5.4} \approx 251,189.
\]
In other words, the airplane taking off produces sound about 250 thousand times more intense than your normal speech.

(b) Each increase of 10 decibels results in a doubling of loudness. Here we have an increase of 54 decibels, so we have had an increase of 10 decibels 5.4 times. Thus the perceived loudness has increased by a factor of \(2^{5.4}\). Because \(2^{5.4} \approx 42\), this means that the airplane seems about 42 times louder than your normal speech.

47 Suppose a television is playing softly at a sound level of 50 decibels. What decibel level would make the television sound eight times as loud?

**SOLUTION** Each increase of ten decibels makes the television sound twice as loud. Because \(8 = 2^3\), the sound level must double three times to make the television sound eight times as loud. Thus 30 decibels must be added to the sound level, raising it to 80 decibels.

49 Suppose a motorcycle produces a sound level of 90 decibels. What decibel level would make the motorcycle sound one-third as loud?

**SOLUTION** Each decrease of ten decibels makes the motorcycle sound half as loud. The sound level must be cut in half \(x\) times, where \(\frac{1}{2} = (\frac{1}{2})^x\), to make the motorcycle sound one-third as loud. This equation can be rewritten as \(2^x = 3\). Taking common logarithms of both sides gives \(x \log 2 = \log 3\), which implies that
\[
x = \frac{\log 3}{\log 2} \approx 1.583.
\]
Thus the sound level must be decreased by ten decibels 1.585 times, meaning that the sound level must be reduced by 15.85 decibels. Because \(90 - 15.85 = 74.15\), a sound level of 74.15 decibels would make the motorcycle sound one-third as loud.

51 How many times brighter is a star with apparent magnitude 2 than a star with apparent magnitude 17?

**SOLUTION** Every five magnitudes correspond to a change in brightness by a factor of 100. Thus a change in 15 magnitudes corresponds to a change in brightness by a factor of \(100^{15}\) (because \(15 = 5 \times 3\)). Because \(100^3 = (10^2)^3 = 10^6\), a star with apparent magnitude 2 is one million times brighter than a star with apparent magnitude 17.

53 Sirion, the brightest star that can be seen from Earth (not counting the sun), has an apparent magnitude of \(-1.4\). Vega, which was the North Star about 12,000 years ago (slight changes in Earth’s orbit lead to changing North Stars every several thousand years), has an apparent magnitude of 0.03. How many times brighter than Vega is Sirius?

**SOLUTION** Let \(b_V\) denote the brightness of Vega and let \(b_S\) denote the brightness of Sirius. Thus
\[
0.03 = \frac{5}{2} \log \frac{b_0}{b_V} \quad \text{and} \quad -1.4 = \frac{5}{2} \log \frac{b_0}{b_S}.
\]
Subtracting the second equation from the first equation, we get
\[
1.43 = \frac{5}{2} \log \frac{b_0}{b_V} - \frac{5}{2} \log \frac{b_0}{b_S}.
\]
Multiplying both sides by \(\frac{2}{5}\), we get
\[
0.572 = \log \frac{b_0}{b_V} - \log \frac{b_0}{b_S} = \log \left( \frac{b_0}{b_V} / \frac{b_0}{b_S} \right) = \log \frac{b_S}{b_V}.
\]
Thus
\[
\frac{b_S}{b_V} = 10^{0.572} \approx 3.7.
\]
Thus Sirius is approximately 3.7 times brighter than Vega.

55 Neptune has an apparent magnitude of about 7.8. What is the apparent magnitude of a star that is 20 times brighter than Neptune?

**SOLUTION** Each decrease of apparent magnitude by 1 corresponds to brightness increase by a factor of \(100^{1/5}\). If we decrease the magnitude by \(x\), then the brightness increases by a factor of \((100^{1/5})^x\). For this exercise, we want \(20 = (100^{1/5})^x\). To solve this equation for \(x\), take logarithms of both sides, getting
\[
\log 20 = x \log(100^{1/5}) = \frac{2x}{5}.
\]
Thus
\[
x = \frac{5}{2} \log 20 \approx 3.25.
\]
Because \(7.8 - 3.25 = 4.55\), we conclude that a star 20 times brighter than Neptune has apparent magnitude approximately 4.55.
3.4 Exponential Growth

LEARNING OBJECTIVES

By the end of this section you should be able to
- describe the behavior of functions with exponential growth;
- model population growth;
- compute compound interest.

We begin this section with a story.

A Doubling Fable

A mathematician in ancient India invented the game of chess. Filled with gratitude for the remarkable entertainment of this game, the King offered the mathematician anything he wanted. The King expected the mathematician to ask for rare jewels or a majestic palace.

But the mathematician asked only that he be given one grain of rice for the first square on a chessboard, plus two grains of rice for the next square, plus four grains for the next square, and so on, doubling the amount for each square, until the 64th square on an 8-by-8 chessboard had been reached. The King was pleasantly surprised that the mathematician had asked for such a modest reward.

A bag of rice was opened, and first 1 grain was set aside, then 2, then 4, then 8, and so on. As the eighth square (the end of the first row of the chessboard) was reached, 128 grains of rice were counted out. The King was secretly delighted to be paying such a small reward and also wondering at the foolishness of the mathematician.

As the 16th square was reached, 32,768 grains of rice were counted out, but this was still a small part of a bag of rice. But the 21st square required a full bag of rice, and the 24th square required eight bags of rice. This was more than the King had expected, but it was a trivial amount because the royal granary contained about 200,000 bags of rice to feed the kingdom during the coming winter.

As the 31st square was reached, over a thousand bags of rice were required and were delivered from the royal granary. Now the King was worried. By the 37th square, the royal granary was two-thirds empty. The 38th square would have required more bags of rice than were left, but the King stopped the process and ordered that the mathematician’s head be chopped off as a warning about the greed induced by exponential growth.

<table>
<thead>
<tr>
<th>n</th>
<th>$2^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1024</td>
</tr>
<tr>
<td>20</td>
<td>1048576</td>
</tr>
<tr>
<td>30</td>
<td>1073741824</td>
</tr>
<tr>
<td>40</td>
<td>1099511627776</td>
</tr>
<tr>
<td>50</td>
<td>1125899906842624</td>
</tr>
<tr>
<td>60</td>
<td>1152921504606846976</td>
</tr>
</tbody>
</table>

To understand why the mathematician’s seemingly modest request turned out to be so extravagant, note that the $n$th square of the chessboard required $2^{n-1}$ grains of rice. These numbers start slowly but grow rapidly, as shown in the table here.
The 64th square of the chessboard would have required $2^{63}$ grains of rice. To estimate of the magnitude of this number, note that $2^{10} = 1024 \approx 10^3$. Thus

$$2^{63} = 2^3 \cdot 2^{60} = 8 \cdot (2^{10})^6 \approx 8 \cdot (10^3)^6 = 8 \cdot 10^{18} \approx 10^{19}.$$  

If each large bag contains a million (which equals $10^6$) grains of rice, then the approximately $10^{19}$ grains of rice needed for the 64th square would have required approximately $10^{19}/10^6$ bags of rice, or approximately $10^{13}$ bags of rice. If we assume ancient India had a population of about ten million ($10^7$), then each resident would have had to produce about $10^{13}/10^7$ bags of rice to satisfy the mathematician’s request for the 64th square of the chessboard. It would have been impossible for each resident in India to produce a million ($10^{13}/10^7$) bags of rice. Thus the mathematician should not have been surprised at losing his head.

### Functions with Exponential Growth

The function $f$ defined by $f(x) = 2^x$ is an example of what is called a function with exponential growth. Other examples of functions with exponential growth are the functions $g$ and $h$ defined by $g(x) = 3 \cdot 5^x$ and $h(x) = 5 \cdot 7^{3x}$. More generally, we have the following definition:

**Exponential growth**

A function $f$ is said to have **exponential growth** if $f$ is of the form

$$f(x) = cb^{kx},$$

where $c$ and $k$ are positive numbers and $b > 1$.

Functions with exponential growth increase rapidly. In fact, every function with exponential growth increases more rapidly than every polynomial, in the sense that if $f$ is a function with exponential growth and $p$ is any polynomial, then $f(x) > p(x)$ for all sufficiently large $x$. For example, $2^x > x^{1000}$ for all $x > 13747$ (Problem 35 shows that 13747 could not be replaced by 13746).

Functions with exponential growth increase so rapidly that graphing them in the usual manner can display too little information, as shown in the following example.

Discuss the graph of the function $9^x$ on the interval $[0, 8]$.

**SOLUTION** The graph of the function $9^x$ on the interval $[0, 8]$ is shown here. In this graph, we cannot use the same scale on the horizontal and vertical axes because $9^8$ is larger than forty million.

Due to the scale, this graph of $9^x$ is hard to distinguish from the horizontal axis in the interval $[0, 5]$. Thus this graph gives little insight into the behavior of the function there. For example, this graph does not adequately distinguish between the values $9^2$ (which equals 81) and $9^5$ (which equals 59049).
Because the graphs of functions with exponential growth often do not provide sufficient visual information, data that is expected to have exponential growth is often graphed by taking the logarithm of the data. The advantage of this procedure is that if \( f \) is a function with exponential growth, then the logarithm of \( f \) is a linear function. For example, if

\[
f(x) = 2 \cdot 3^{5x},
\]

then

\[
\log f(x) = 5 \log 3 \cdot x + \log 2;
\]

thus the graph of \( \log f \) is the line whose equation is \( y = 5 \log 3 \cdot x + \log 2 \) (which is the line with slope \( 5 \log 3 \)).

More generally, if \( f(x) = c b^{kx} \), then

\[
\log f(x) = \log c + \log(b^{kx})
= k \log b \cdot x + \log c.
\]

Here \( k, \log b, \) and \( \log c \) are all numbers not dependent on \( x \); thus the function \( \log f \) is indeed linear. If \( k > 0 \) and \( b > 1 \), as is required in the definition of exponential growth, then \( k \log b > 0 \), which implies that the line \( y = \log f(x) \) has positive slope.

---

**Logarithm of a function with exponential growth**

A function \( f \) has exponential growth if and only if the graph of \( \log f(x) \) is a line with positive slope.

---

**Example 2**

Moore's Law is the phrase used to describe the observation that computing power has exponential growth, doubling roughly every 18 months. One standard measure of computing power is the number of transistors used per integrated circuit. The logarithm of this quantity (for common computer chips manufactured by Intel) is shown in the graph below for certain years between 1972 and 2010, with line segments connecting the data points:

The logarithm of the number of transistors per integrated circuit. Moore's Law predicts exponential growth of computing power, which would make this graph a line.

Does this graph indicate that computing power has had exponential growth?
SOLUTION The graph of the logarithm of the number of transistors is roughly a line, as would be expected for a function with roughly exponential growth. Thus the graph does indeed indicate that computing power has had exponential growth.

Real data, as shown in the graph in this example, rarely fits theoretical mathematical models perfectly. The graph above is not exactly a line, and thus we do not have exactly exponential growth. However, the graph above is close enough to a line so that a model of exponential growth can help explain what has happened to computing power over several decades.

Consider the function $f$ with exponential growth defined by

$$f(x) = 5 \cdot 3^{2x}.$$ 

Because $3^{2x} = (3^2)^x = 9^x$, we can rewrite $f$ in the form

$$f(x) = 5 \cdot 9^x.$$ 

More generally, suppose $f$ is a function with exponential growth defined by

$$f(x) = c B^{kx}.$$ 

Because $B^{kx} = (B^k)^x$, if we let $b = B^k$ then we can rewrite $f$ in the form

$$f(x) = c b^x.$$ 

In other words, by changing $b$ we can, if we wish, always take $k = 1$ in the definition given earlier of a function with exponential growth.

**Exponential growth, simpler form**

Every function $f$ with exponential growth can be written in the form

$$f(x) = c b^x,$$

where $c > 0$ and $b > 1$.

Consider now the function $f$ with exponential growth defined by

$$f(x) = 3 \cdot 5^{7x}.$$ 

Because $5^{7x} = (2^{\log_2 5})^{7x} = 2^{7(\log_2 5)x}$, we can rewrite $f$ in the form

$$f(x) = 3 \cdot 2^{kx},$$

where $k = 7(\log_2 5)$.

There is nothing special about the numbers 5 and 7 that appear in the paragraph above. The same procedure could be applied to any function $f$ with exponential growth defined by $f(x) = c b^{kx}$. Thus we have the following result, which shows that by changing $k$ we can, if we wish, always take $b = 2$ in the definition given earlier of a function with exponential growth.
Exponential Functions, Logarithms, and \( e \)

**Exponential growth, base 2**

Every function \( f \) with exponential growth can be written in the form

\[
f(x) = c2^{kx},
\]

where \( c \) and \( k \) are positive numbers.

In the result above, there is nothing special about the number 2. The same result holds if 2 is replaced by 3 or 4 or any number bigger than 1. In other words, we can choose the base for a function with exponential growth to be whatever we wish (and then \( k \) needs to be suitably adjusted). You will often want to choose a value for the base that is related to the topic under consideration. We will soon consider population doubling models, where 2 is the most natural choice for the base.

---

**EXAMPLE 3**

Suppose \( f \) is a function with exponential growth such that \( f(2) = 3 \) and \( f(5) = 7 \).

(a) Find a formula for \( f(x) \).

(b) Evaluate \( f(17) \).

**SOLUTION**

(a) We will use the simpler form derived above. In other words, we can assume

\[
f(x) = cb^x.
\]

We need to find \( c \) and \( b \). We have

\[
3 = f(2) = cb^2 \quad \text{and} \quad 7 = f(5) = cb^5.
\]

Dividing the second equation by the first equation shows that \( b^3 = \frac{7}{3} \). Thus \( b = \left(\frac{7}{3}\right)^{\frac{1}{3}} \). Substituting this value for \( b \) into the first equation above gives

\[
3 = c\left(\frac{7}{3}\right)^{\frac{2}{3}},
\]

which implies that \( c = 3\left(\frac{7}{3}\right)^{\frac{2}{3}} \). Thus

\[
f(x) = 3\left(\frac{7}{3}\right)^{\frac{2}{3}} \left(\frac{7}{3}\right)^{x/3}.
\]

(b) Using the formula above, we have

\[
f(17) = 3\left(\frac{7}{3}\right)^{\frac{2}{3}} \left(\frac{7}{3}\right)^{17/3}
\]

\[
= 207.494.
\]
Population Growth

Populations of various organisms, ranging from bacteria to humans, often exhibit exponential growth. To illustrate this behavior, we will begin by considering bacteria. Bacteria are single-celled creatures that reproduce by absorbing some nutrients, growing, and then dividing in half—one bacterium cell becomes two bacteria cells.

Suppose a colony of bacteria in a petri dish has 700 cells at 1 PM. These bacteria reproduce at a rate that leads to doubling every three hours. How many bacteria cells will be in the petri dish at 9 PM on the same day?

**SOLUTION**  Because the number of bacteria cells doubles every three hours, at 4 PM there will be 1400 cells, at 7 PM there will be 2800 cells, and so on. In other words, in three hours the number of cells increases by a factor of two, in six hours the number of cells increases by a factor of four, in nine hours the number of cells increases by a factor of eight, and so on.

More generally, in $t$ hours there are $t/3$ doubling periods. Hence in $t$ hours the number of cells increases by a factor of $2^{t/3}$ and we should have

$$700 \cdot 2^{t/3}$$

bacteria cells.

Thus at 9 PM, which is eight hours after 1 PM, our colony of bacteria should have $700 \cdot 2^{8/3}$ cells. However, this result should be thought of as an estimate rather than as an exact count. Actually, $700 \cdot 2^{8/3}$ is an irrational number (approximately 4444.7), which makes no sense when counting bacteria cells. Thus we might predict that at 9 PM there would be about 4445 cells. Even better, because the real world rarely strictly adheres to formulas, we might expect between 4400 and 4500 cells at 9 PM.

Although a function with exponential growth will often provide the best model for population growth for a certain time period, real population data cannot exhibit exponential growth for excessively long time periods. For example, the formula $700 \cdot 2^{t/3}$ derived above for our colony of bacteria predicts that after 10 days, which equals 240 hours, we would have about $10^{27}$ cells, which is far more than could fit in even a gigantic petri dish. The bacteria would have run out of space and nutrients long before reaching this population level.

Now we extend our example with bacteria to a more general situation. Suppose a population doubles every $d$ time units (here the time units might be hours, days, years, or whatever unit is appropriate). Suppose also that at some specific time $t_0$ we know that the population is $p_0$. At time $t$ there have been $t - t_0$ time units since time $t_0$. Thus at time $t$ there have been $(t - t_0)/d$ doubling periods, and hence the population increases by a factor of

$$2^{(t - t_0)/d}.$$ 

This factor must be multiplied by the population at the starting time $t_0$. In other words, at time $t$ we could expect a population of $p_0 \cdot 2^{(t - t_0)/d}$. 

**EXAMPLE 4**

A bacterium cell dividing, as photographed by an electron microscope.

Because functions with exponential growth increase so rapidly, they can be used to model real data for only limited time periods.
**Exponential growth and doubling**

If a population doubles every \( d \) time units, then the function \( p \) modeling this population growth is given by the formula

\[
p(t) = p_0 \cdot 2^{(t-t_0)/d},
\]

where \( p_0 \) is the population at time \( t_0 \).

The function \( p \) has exponential growth because we could rewrite \( p \) in the form

\[
p(t) = (2^{-t_0/d} p_0) 2^{(1/d) t},
\]

which corresponds to our definition of a function with exponential growth by taking \( c = 2^{-t_0/d} p_0, b = 2, \) and \( k = 1/d \).

Human population data often follow patterns of exponential growth for decades or centuries. The graph below shows the logarithm of the world population for each year from 1950 to 2000:

![Graph of the logarithm of world population](image)

The logarithm of the world population each year from 1950 to 2000, as estimated by the U.S. Census Bureau.

**EXAMPLE 5**

World population is now increasing at a slower rate, doubling about every 69 years.

The world population in mid-year 1950 was about 2.56 billion. During the period 1950-2000, world population increased at a rate that doubled the population approximately every 40 years.

(a) Find a formula that estimates the mid-year world population for 1950-2000.

(b) Using the formula from part (a), estimate the world population in mid-year 1955.

**SOLUTION**

(a) Using the formula and data above, we see that the mid-year world population in the year \( y \), expressed in billions, was approximately

\[
2.56 \cdot 2^{(y-1950)/40}.
\]

(b) Taking \( y = 1955 \) in the formula above gives the estimate that the mid-year world population in 1955 was \( 2.56 \cdot 2^{(1955-1950)/40} \) billion, which is approximately 2.79 billion. The actual value was about 2.78 billion; thus the formula has good accuracy in this case.


### Compound Interest

This Dilbert comic illustrates the power of compound interest. See the solution to Exercise 33 to learn whether this plan would work.

The computation of compound interest involves functions with exponential growth. We begin with a simple example.

**Example 6**

Suppose you deposit $8000 in a bank account that pays 5% annual interest. Assume the bank pays interest once per year at the end of the year, and that each year you place the interest in a cookie jar for safekeeping.

(a) How much will you have (original amount plus interest) at the end of two years?

(b) How much will you have (original amount plus interest) at the end of $t$ years?

**Solution**

(a) Because 5% of $8000 is $400, at the end of the each year you will receive $400 interest. Thus after two years you will have $800 in the cookie jar, bringing the total amount to $8800.

(b) Because you receive $400 interest each year, at the end of $t$ years the cookie jar will contain $400t$ dollars. Thus the total amount you will have at the end of $t$ years is $8000 + 400t$ dollars.

The situation in the example above, where interest is paid only on the original amount, is called **simple interest**. To generalize the example above, we can replace the $8000 used in the example above with any initial amount $P$. Furthermore, we can replace the 5% annual interest with any annual interest rate $r$, expressed as a number rather than as a percent (thus 5% interest would correspond to $r = 0.05$). Each year the interest received will be $rP$. Thus after $t$ years the total interest received will be $rPt$. Hence the total amount after $t$ years will be $P + rPt$. Factoring out $P$ from this expression, we have the following result:

The symbol $P$ comes from **principal**, which is a fancy word for the initial amount.
Simple interest

If interest is paid once per year at annual interest rate \( r \), with no interest paid on the interest, then after \( t \) years an initial amount \( P \) grows to

\[
P(1 + rt).
\]

The expression \( P(1 + rt) \) above is a linear function of \( t \) (assuming that the principal \( P \) and the interest rate \( r \) are fixed). Thus when money grows with simple interest, linear functions arise naturally. We now turn to the more realistic situation of compound interest, meaning that interest is paid on the interest.

**EXAMPLE 7**

Suppose you deposit $8000 in a bank account that pays 5% annual interest. Assume the bank pays interest once per year at the end of the year, and that each year the interest is deposited in the bank account.

(a) How much will you have at the end of one year?

(b) How much will you have at the end of two years?

(c) How much will you have at the end of \( t \) years?

**SOLUTION**

(a) Because 5% of $8000 is $400, at the end of the first year you will receive $400 interest. Thus at the end of the first year the bank account will contain $8400.

(b) At the end of the second year you will receive as interest 5% of $8400, which equals $420, which when added to the bank account gives a total of $8820.

(c) Note that each year the amount in the bank account increases by a factor of 1.05. At the end of the first year you will have

\[
8000 \times 1.05
\]

dollars (which equals $8400). At the end of two years, you will have the amount above multiplied by 1.05, which equals

\[
8000 \times 1.05^2
\]

dollars (which equals $8820). At the end of three years, you will have the amount above again multiplied by 1.05, which equals

\[
8000 \times 1.05^3
\]

dollars (which equals $9261). After \( t \) years, the original $8000 will have grown to

\[
8000 \times 1.05^t
\]
dollars.
The table below summarizes the data for the two methods of computing interest that we have considered in the last two examples.

<table>
<thead>
<tr>
<th>year</th>
<th>simple interest</th>
<th>compound interest</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>interest</td>
<td></td>
</tr>
<tr>
<td></td>
<td>total</td>
<td>interest</td>
</tr>
<tr>
<td></td>
<td>total</td>
<td></td>
</tr>
<tr>
<td>initial amount</td>
<td>$8000</td>
<td>$8000</td>
</tr>
<tr>
<td>1</td>
<td>$400</td>
<td>$8400</td>
</tr>
<tr>
<td>2</td>
<td>$400</td>
<td>$8800</td>
</tr>
<tr>
<td>3</td>
<td>$400</td>
<td>$9200</td>
</tr>
</tbody>
</table>

*Simple and compound interest, once per year, on $8000 at 5%.*

To generalize the example above, we can replace the $8000 used above with any initial amount $P$. Furthermore, we can replace the 5% annual interest with any annual interest rate $r$, expressed as a number rather than as a percent. Each year the amount in the bank account increases by a factor of $1 + r$. Thus at the end of the first year the initial amount $P$ will grow to $P(1 + r)$. At the end of two years, this will have grown to $P(1 + r)^2$. At the end of three years, this will have grown to $P(1 + r)^3$. More generally, we have the following result:

**Compound interest, once per year**

If interest is compounded once per year at annual interest rate $r$, then after $t$ years an initial amount $P$ grows to

$$P(1 + r)^t.$$

The expression $P(1 + r)^t$ above has exponential growth as a function of $t$. Because functions with exponential growth increase rapidly, compound interest can lead to large amounts of money after long time periods.

In 1626 Dutch settlers supposedly purchased from Native Americans the island of Manhattan for $24. To determine whether or not this was a bargain, suppose $24 earned 7% per year (a reasonable rate for a real estate investment), compounded once per year since 1626. How much would this investment be worth by 2012?

**EXAMPLE 8**

Little historical evidence exists concerning the alleged sale of Manhattan. Most of the stories about this event should be considered legends.

Today Manhattan contains well-known landmarks such as Times Square, the Empire State Building, Wall Street, and United Nations headquarters.

**SOLUTION**

Because $2012 - 1626 = 386$, the formula above shows that an initial amount of $24 earning 7% per year compounded once per year would be worth

$$24(1.07)^{386}$$

dollars in 2012. A calculator shows that this is over five trillion dollars, which is more than the current assessed value of all land in Manhattan.
Interest is often compounded more than once per year. To see how this works, we now modify an earlier example. In our new example, interest will be paid and compounded twice per year rather than once per year. This means that instead of 5% interest being paid at the end of each year, the interest comes as two payments of 2.5% each year, with the 2.5% interest payments made at the end of every six months.

**Example 9**

Suppose you deposit $8000 in a bank account that pays 5% annual interest, compounded twice per year. How much will you have at the end of one year?

**Solution**

Because 2.5% of $8000 equals $200, at the end of the first six months $200 will be deposited into the bank account; the bank account will then have a total of $8200.

At the end of the second six months (in other words, at the end of the first year), 2.5% interest will be paid on the $8200 that was in the bank account for the previous six months. Because 2.5% of $8200 equals $205, the bank account will have $8405 at the end of the first year.

In the example above, the $8405 in the bank account at the end of the first year should be compared with the $8400 that would be in the bank account if interest had been paid only at the end of the year. The extra $5 arises because of the interest during the second six months on the interest earned at the end of the first six months.

Instead of compounding interest twice per year, as in the previous example, interest could be compounded four times per year. At 5% annual interest, this would mean that 1.25% interest will be paid at the end of every three months. Or interest could be compounded 12 times per year, with \( \frac{5}{12} \% \) interest at the end of each month. The table below shows the growth of $8000 at 5% interest for three years, with compounding either 1, 2, 4, or 12 times per year.

<table>
<thead>
<tr>
<th>year</th>
<th>times compounded per year</th>
<th>The growth of $8000 at 5% interest, rounded to the nearest dollar.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>initial amount</td>
<td>$8000</td>
<td>$8000</td>
</tr>
<tr>
<td>1</td>
<td>$8400</td>
<td>$8405</td>
</tr>
<tr>
<td>2</td>
<td>$8820</td>
<td>$8831</td>
</tr>
<tr>
<td>3</td>
<td>$9261</td>
<td>$9278</td>
</tr>
</tbody>
</table>

To find a formula for how money grows when compounded more than once per year, consider a bank account with annual interest rate \( r \), compounded twice per year. Thus every six months, the amount in the bank account increases by a factor of \( 1 + \frac{r}{2} \). After \( t \) years, this will happen \( 2t \) times. Thus an initial amount \( P \) will grow to \( P \left(1 + \frac{r}{2}\right)^{2t} \) in \( t \) years.

More generally, suppose now that an annual interest rate \( r \) is compounded \( n \) times per year. Then \( n \) times per year, the amount in the bank account increases by a factor of \( 1 + \frac{r}{n} \). After \( t \) years, this will happen \( nt \) times, leading to the following result:
Compound interest, \( n \) times per year

If interest is compounded \( n \) times per year at annual interest rate \( r \), then after \( t \) years an initial amount \( P \) grows to

\[
P(1 + \frac{r}{n})^{nt}.
\]

Suppose a bank account starting out with $8000 receives 5\% annual interest, compounded twelve times per year. How much will be in the bank account after three years?

**SOLUTION**  Take \( r = 0.05 \), \( n = 12 \), \( t = 3 \), and \( P = 8000 \) in the formula above, which shows that after three years the amount in the bank account will be

\[
8000(1 + \frac{0.05}{12})^{12 \cdot 3}
\]

dollars. A calculator shows that this amount is approximately $9292 (which is the last entry in the table above).

Advertisements from financial institutions often list the “APY” that you will earn on your money rather than the interest rate. The abbreviation “APY” denotes “annual percentage yield”, which means the actual interest rate that you would receive at the end of one year after compounding.

For example, if a bank is paying 5\% annual interest, compounded once per month (as is fairly common), then the bank can legally advertise that it pays an APY of 5.116\%. Here the APY equals 5.116\% because

\[
(1 + \frac{0.05}{12})^{12} \approx 1.05116.
\]

In other words, at 5\% annual interest compounded twelve times per year, $1000 will grow to $1051.16. For a period of one year, this corresponds to simple annual interest of 5.116\%.

**EXERCISES**

1. Without using a calculator or computer, give a rough estimate of \( 2^{83} \).
2. Without using a calculator or computer, give a rough estimate of \( 2^{103} \).
3. Without using a calculator or computer, determine which of the two numbers \( 2^{125} \) and \( 32 \cdot 10^{36} \) is larger.
4. Without using a calculator or computer, determine which of the two numbers \( 2^{400} \) and \( 17^{100} \) is larger.  
   [Hint: Note that \( 2^4 = 16 \).]

For Exercises 5–8, suppose you deposit into a savings account one cent on January 1, two cents on January 2, four cents on January 3, and so on, doubling the amount of your deposit each day (assume you use an electronic bank that is open every day of the year).

5. How much will you deposit on January 7?
6. How much will you deposit on January 11?
7. What is the first day that your deposit will exceed $10,000?
8. What is the first day that your deposit will exceed $100,000?
For Exercises 9–12, suppose you deposit into your savings account one cent on January 1, three cents on January 2, nine cents on January 3, and so on, tripling the amount of your deposit each day.

9. How much will you deposit on January 7?
10. How much will you deposit on January 11?
11. What is the first day that your deposit will exceed $10,000?
12. What is the first day that your deposit will exceed $100,000?

13. Suppose \( f(x) = 7 \cdot 2^{3x} \). Find a number \( b \) such that the graph of \( \log_b f \) has slope 1.
14. Suppose \( f(x) = 4 \cdot 2^{5x} \). Find a number \( b \) such that the graph of \( \log_b f \) has slope 1.

15. A colony of bacteria is growing exponentially, doubling in size every 100 minutes. How many minutes will it take for the colony of bacteria to triple in size?
16. A colony of bacteria is growing exponentially, doubling in size every 140 minutes. How many minutes will it take for the colony of bacteria to become five times its current size?

17. At current growth rates, the Earth’s population is doubling about every 69 years. If this growth rate were to continue, about how many years will it take for the Earth’s population to increase 50% from the present level?
18. At current growth rates, the Earth's population is doubling about every 69 years. If this growth rate were to continue, about how many years will it take for the Earth’s population to become one-fourth larger than the current level?

19. Suppose a colony of bacteria starts with 200 cells and triples in size every four hours.
   (a) Find a function that models the population growth of this colony of bacteria.
   (b) Approximately how many cells will be in the colony after six hours?

20. Suppose a colony of bacteria starts with 100 cells and triples in size every two hours.
   (a) Find a function that models the population growth of this colony of bacteria.
   (b) Approximately how many cells will be in the colony after one hour?

21. Suppose $700 is deposited in a bank account paying 6% interest per year, compounded 52 times per year. How much will be in the bank account at the end of 10 years?

22. Suppose $8000 is deposited in a bank account paying 7% interest per year, compounded 12 times per year. How much will be in the bank account at the end of 100 years?

23. Suppose a bank account paying 4% interest per year, compounded 12 times per year, contains $10,555 at the end of 10 years. What was the initial amount deposited in the bank account?

24. Suppose a bank account paying 6% interest per year, compounded four times per year, contains $27,707 at the end of 20 years. What was the initial amount deposited in the bank account?

25. Suppose a savings account pays 6% interest per year, compounded once per year. If the savings account starts with $500, how long would it take for the savings account to exceed $2000?

26. Suppose a savings account pays 5% interest per year, compounded four times per year. If the savings account starts with $600, how many years would it take for the savings account to exceed $1400?

27. Suppose a bank wants to advertise that $1000 deposited in its savings account will grow to $1040 in one year. This bank compounds interest 12 times per year. What annual interest rate must the bank pay?

28. Suppose a bank wants to advertise that $1000 deposited in its savings account will grow to $1050 in one year. This bank compounds interest 365 times per year. What annual interest rate must the bank pay?


   Did you know that the percent increase of the value of a home in Manhattan between the years 1950 and 2000 was 721%? Buy a home in Manhattan and invest in your future.

   Suppose that instead of buying a home in Manhattan in 1950, someone had invested money in a bank account that compounds interest four times per year. What annual interest rate would the bank have to pay to equal the growth claimed in the ad?

30. Suppose that instead of buying a home in Manhattan in 1950, someone had invested money in a bank account that compounds interest once per month. What annual interest rate would the bank have to pay to equal the growth claimed in the ad from the previous exercise?
31 Suppose \( f \) is a function with exponential growth such that
\[ f(1) = 3 \quad \text{and} \quad f(3) = 5. \]
Evaluate \( f(8) \).

32 Suppose \( f \) is a function with exponential growth such that
\[ f(2) = 3 \quad \text{and} \quad f(5) = 8. \]
Evaluate \( f(10) \).

PROBLEMS

35 Explain how you would use a calculator to verify that
\[ 2^{13746} < 13746^{1000} \]
but
\[ 2^{13747} > 13747^{1000}, \]
and then actually use a calculator to verify both these inequalities.
[The numbers involved in these inequalities have over four thousand digits. Thus some cleverness in using your calculator is required.]

36 Show that
\[ 2^{10n} = (1.024)^n 10^{3n}. \]
[This equality leads to the approximation \( 2^{10n} \approx 10^{3n} \).]

37 Show that if \( f \) is a function with exponential growth, then so is the square root of \( f \). More precisely, show that if \( f \) is a function with exponential growth, then so is the function \( g \) defined by \( g(x) = \sqrt{f(x)} \).

38 Suppose \( f \) is a function with exponential growth and \( f(0) = 1 \). Explain why \( f \) can be represented by a formula of the form \( f(x) = b^x \) for some \( b > 1 \).

39 Explain why every function \( f \) with exponential growth can be represented by a formula of the form \( f(x) = c \cdot 3^{kx} \) for appropriate choices of \( c \) and \( k \).

40 Find three newspaper articles that use the word “exponentially” (one way to do this is to use the web site of a newspaper that allows searches). For each use of the word “exponentially” that you find in a newspaper article, discuss whether or not the word is used in its correct mathematical sense.

41 Suppose a bank pays annual interest rate \( r \), compounded \( n \) times per year. Explain why the bank can advertise that its APY equals
\[ (1 + \frac{r}{n})^n - 1. \]

Exercises 33–34 will help you determine whether or not the Dilbert comic earlier in this section gives a reasonable method for turning a hundred dollars into a million dollars.

33 At 5% interest compounded once per year, how many years will it take to turn a hundred dollars into a million dollars?

34 At 5% interest compounded monthly, how long will it take to turn a hundred dollars into a million dollars?

42 Find an advertisement in a newspaper or web site that gives the interest rate (before compounding), the frequency of compounding, and the APY. Determine whether or not the APY has been computed correctly.

43 Suppose \( f \) is a function with exponential growth. Show that there is a number \( b > 1 \) such that
\[ f(x + 1) = bf(x) \]
for every \( x \).

44 What is wrong with the following apparent paradox: You have two parents, four grandparents, eight great-grandparents, and so in. Going back \( n \) generations, you should have \( 2^n \) ancestors. Assuming three generations per century, if we go back 2000 years (which equals 20 centuries and thus 60 generations), then you should have \( 2^{60} \) ancestors from 2000 years ago. However, \( 2^{60} = (2^{10})^6 \approx (10^3)^6 = 10^{18} \), which equals a billion billion, which is far more than the total number of people who have ever lived.
WORKED-OUT SOLUTIONS to Odd-Numbered Exercises

1. Without using a calculator or computer, give a rough estimate of $2^{83}$.

**SOLUTION**

$$2^{83} = 2^3 \cdot 2^{80} = 8 \cdot 2^{10 \cdot 8} = 8 \cdot (2^{10})^8$$

$$\approx 8 \cdot (10^8)^8 = 8 \cdot 10^{24} \approx 10^{25}$$

2. Without using a calculator or computer, determine which of the two numbers $2^{125}$ and $32 \cdot 10^{36}$ is larger.

**SOLUTION**

Note that

$$2^{125} = 2^5 \cdot 2^{120}$$

$$= 32 \cdot (2^{10})^{12}$$

$$> 32 \cdot (10^4)^{12}$$

$$= 32 \cdot 10^{36}.$$  

Thus $2^{125}$ is larger than $32 \cdot 10^{36}.

For Exercises 5–8, suppose you deposit into a savings account one cent on January 1, two cents on January 2, four cents on January 3, and so on, doubling the amount of your deposit each day (assume you use an electronic bank that is open every day of the year).

5. How much will you deposit on January 7?

**SOLUTION**

On the $n$th day, $2^{n-1}$ cents are deposited. Thus on January 7, the amount deposited is $2^6$ cents. In other words, $0.64$ will be deposited on January 7.

7. What is the first day that your deposit will exceed $10,000$?

**SOLUTION**

On the $n$th day, $2^{n-1}$ cents are deposited. Because $10,000$ equals $10^6$ cents, we need to find the smallest integer $n$ such that

$$2^{n-1} > 10^6.$$  

We can do a quick estimate by noting that

$$10^6 = (10^3)^2 < (2^{10})^2 = 2^{20}.$$  

Thus taking $n - 1 = 20$, which is equivalent to taking $n = 21$, should be close to the correct answer.

To be more precise, note that the inequality $2^{n-1} > 10^6$ is equivalent to the inequality

$$\log(2^{n-1}) > \log(10^6),$$

which can be rewritten as

$$(n - 1) \log 2 > 6.$$  

Dividing both sides by $\log 2$ and then adding $1$ to both sides shows that this is equivalent to

$$n > 1 + \frac{6}{\log 2}.$$  

A calculator shows that $1 + \frac{6}{\log 2} \approx 20.9$. Because $21$ is the smallest integer bigger than $20.9$, January 21 is the first day that the deposit will exceed $10,000$.

For Exercises 9–12, suppose you deposit into your savings account one cent on January 1, three cents on January 2, nine cents on January 3, and so on, tripling the amount of your deposit each day.

9. How much will you deposit on January 7?

**SOLUTION**

On the $n$th day, $3^{n-1}$ cents are deposited. Thus on January 7, the amount deposited is $3^6$ cents. Because $3^6 = 729$, we conclude that $7.29$ will be deposited on January 7.

11. What is the first day that your deposit will exceed $10,000$?

**SOLUTION**

On the $n$th day, $3^{n-1}$ cents are deposited. Because $10,000$ equals $10^6$ cents, we need to find the smallest integer $n$ such that

$$3^{n-1} > 10^6.$$  

This is equivalent to the inequality

$$\log(3^{n-1}) > \log(10^6),$$

which can be rewritten as

$$(n - 1) \log 3 > 6.$$  

Dividing both sides by $\log 3$ and then adding $1$ to both sides shows that this is equivalent to

$$n > 1 + \frac{6}{\log 3}.$$  

A calculator shows that $1 + \frac{6}{\log 3} \approx 13.6$. Because $14$ is the smallest integer bigger than $13.6$, January 14 is the first day that the deposit will exceed $10,000$.

13. Suppose $f(x) = 7 \cdot 2^{3x}$. Find a number $b$ such that the graph of $\log_b f$ has slope $1$.

**SOLUTION**

Note that

$$\log_b f(x) = \log_b 7 + \log_b (2^{3x})$$

$$= \log_b 7 + 3 \log_b 2 \cdot x.$$  

Thus the slope of the graph of $\log_b f$ equals $3 \log_b 2$, which equals $1$ when $\log_b 2 = \frac{1}{3}$. Thus $b^{1/3} = 2$, which means that $b = 2^3 = 8$.  


A colony of bacteria is growing exponentially, doubling in size every 100 minutes. How many minutes will it take for the colony of bacteria to triple in size?

**SOLUTION** Let \( p(t) \) denote the number of cells in the colony of bacteria at time \( t \), where \( t \) is measured in minutes. Then

\[
p(t) = p_0 2^{t/100},
\]

where \( p_0 \) is the number of cells at time 0. We need to find \( t \) such that \( p(t) = 3p_0 \). In other words, we need to find \( t \) such that

\[
p_0 2^{t/100} = 3p_0.
\]

Dividing both sides of the equation above by \( p_0 \) and then taking the logarithm of both sides gives

\[
\frac{t}{100} \log 2 = \log 3.
\]

Thus \( t = 100 \frac{\log 3}{\log 2} \), which is approximately 158.496. Thus the colony of bacteria will triple in size approximately every 158 minutes.

At current growth rates, the Earth’s population is doubling about every 69 years. If this growth rate were to continue, about how many years will it take for the Earth’s population to increase 50% from the present level?

**SOLUTION** Let \( p(t) \) denote the Earth’s population at time \( t \), where \( t \) is measured in years starting from the present. Then

\[
p(t) = p_0 2^{t/69},
\]

where \( p_0 \) is the present population of the Earth. We need to find \( t \) such that \( p(t) = 1.5p_0 \). In other words, we need to find \( t \) such that

\[
p_0 2^{t/69} = 1.5p_0.
\]

Dividing both sides of the equation above by \( p_0 \) and then taking the logarithm of both sides gives

\[
\frac{t}{69} \log 2 = \log 1.5.
\]

Thus \( t = 69 \frac{\log 1.5}{\log 2} \), which is approximately 40.4. Thus the Earth’s population, at current growth rates, would increase by 50% in approximately 40.4 years.

Suppose a colony of bacteria starts with 200 cells and triples in size every four hours.

(a) Find a function that models the population growth of this colony of bacteria.

(b) Approximately how many cells will be in the colony after six hours?

**SOLUTION**

(a) Let \( p(t) \) denote the number of cells in the colony of bacteria at time \( t \), where \( t \) is measured in hours. We know that \( p(0) = 200 \). In \( t \) hours, there are \( t/4 \) tripling periods; thus the number of cells increases by a factor of \( 3^{t/4} \). Hence

\[
p(t) = 200 \cdot 3^{t/4}.
\]

(b) After six hours, we could expect that there would be \( p(6) \) cells of bacteria. Using the equation above, we have

\[
p(6) = 200 \cdot 3^{6/4} = 200 \cdot 3^{3/2} \approx 1039.
\]

Suppose $700 is deposited in a bank account paying 6% interest per year, compounded 52 times per year. How much will be in the bank account at the end of 10 years?

**SOLUTION** With interest compounded 52 times per year at 6% per year, after 10 years $700 will grow to

\[
700 \left(1 + \frac{0.06}{52}\right)^{52 \cdot 10} \approx $1275.
\]

Suppose a bank account paying 4% interest per year, compounded 12 times per year, contains $10,555 at the end of 10 years. What was the initial amount deposited in the bank account?

**SOLUTION** Let \( P \) denote the initial amount deposited in the bank account. With interest compounded 12 times per year at 4% per year, after 10 years \( P \) dollars will grow to

\[
P \left(1 + \frac{0.04}{12}\right)^{12 \cdot 10} \text{ dollars, which we are told equals } $10,555.
\]

Thus we need to solve the equation

\[
P \left(1 + \frac{0.04}{12}\right)^{120} = $10,555.
\]

The solution to this equation is

\[
P = $10,555 / \left(1 + \frac{0.04}{12}\right)^{120} \approx $7080.
\]

Suppose a savings account pays 6% interest per year, compounded once per year. If the savings account starts with $500, how long would it take for the savings account to exceed $2000?

**SOLUTION** With 6% interest compounded once per year, a savings account starting with $500 would have

\[
500 (1.06)^t \text{ dollars after } t \text{ years. We want this amount to exceed } $2000, \text{ which means that}
\]

\[
500 (1.06)^t > $2000.
\]

The solution to this equation is

\[
t > \log_{1.06} \left(\frac{2000}{500}\right) \approx 22.98.
\]
Chapter 3  Exponential Functions, Logarithms, and e

500(1.06)^t > 2000.

Dividing both sides by 500 and then taking the logarithm of both sides gives

t \log 1.06 > \log 4.

Thus

t > \frac{\log 4}{\log 1.06} \approx 23.8.

Because interest is compounded only once per year, t needs to be an integer. The smallest integer larger than 23.8 is 24. Thus it will take 24 years for the amount in the savings account to exceed $2000.

27 Suppose a bank wants to advertise that $1000 deposited in its savings account will grow to $1040 in one year. This bank compounds interest 12 times per year. What annual interest rate must the bank pay?

SOLUTION Let \( r \) denote the annual interest rate to be paid by the bank. At that interest rate, compounded 12 times per year, in one year $1000 will grow to

\[
1000(1 + \frac{r}{12})^{12}
\]
dollars. We want this to equal $1040, which means that we need to solve the equation

\[
1000(1 + \frac{r}{12})^{12} = 1040.
\]

To solve this equation, divide both sides by 1000 and then raise both sides to the power 1/12, getting

\[
1 + \frac{r}{12} = 1.04^{1/12}.
\]

Now subtract 1 from both sides and then multiply both sides by 12, getting

\[
r = 12(1.04^{1/12} - 1) \approx 0.0393.
\]

Thus the annual interest should be approximately 3.93%.

29 An advertisement for real estate published in the 28 July 2004 New York Times states:

Did you know that the percent increase of the value of a home in Manhattan between the years 1950 and 2000 was 721%? Buy a home in Manhattan and invest in your future.

Suppose that instead of buying a home in Manhattan in 1950, someone had invested money in a bank account that compounds interest four times per year. What annual interest rate would the bank have to pay to equal the growth claimed in the ad?

SOLUTION An increase of 721% means that the final value is 821% of the initial value. Let \( r \) denote the interest rate the bank would have to pay for the 50 years from 1950 to 2000 to grow to 821% of the initial value. At that interest rate, compounded four times per year, in 50 years an initial amount of \( P \) dollars grows to

\[
P(1 + \frac{r}{4})^{4\times50}
\]
dollars. We want this to equal 8.21 times the initial amount, which means that we need to solve the equation

\[
P(1 + \frac{r}{4})^{200} = 8.21P.
\]

To solve this equation, divide both sides by \( P \) and then raise both sides to the power 1/200, getting

\[
1 + \frac{r}{4} = 8.21^{1/200}.
\]

Now subtract 1 from both sides and then multiply both sides by 4, getting

\[
r = 4(8.21^{1/200} - 1) \approx 0.0423.
\]

Thus the annual interest would need to be approximately 4.23% to equal the growth claimed in the ad. [Note that 4.23% is not a particularly high return for a long-term investment, contrary to the ad’s implication.]

31 Suppose \( f \) is a function with exponential growth such that

\[
f(1) = 3 \quad \text{and} \quad f(3) = 5.
\]

Evaluate \( f(8) \).

SOLUTION We can assume

\[
f(x) = cb^x.
\]

We need to find \( c \) and \( b \). We have

\[
3 = f(1) = cb \quad \text{and} \quad 5 = f(3) = cb^3.
\]

Dividing the second equation by the first equation shows that \( b^2 = \frac{5}{3} \). Thus \( b = (\frac{5}{3})^{1/2} \). Substituting this value for \( b \) into the first equation above gives

\[
3 = c(\frac{5}{3})^{1/2},
\]

which implies that \( c = 3(\frac{5}{3})^{1/2} \). Thus

\[
f(x) = 3(\frac{5}{3})^{1/2}(\frac{5}{3})^{x/2}.
\]

Using the formula above, we have

\[
f(8) = 3(\frac{5}{3})^{1/2}(\frac{5}{3})^{4} \approx 625\approx (\frac{5}{3})^{1/2} \approx 17.93.
\]
Exercises 33–34 will help you determine whether or not the Dilbert comic earlier in this section gives a reasonable method for turning a hundred dollars into a million dollars.

33 [At 5% interest compounded once per year, how many years will it take to turn a hundred dollars into a million dollars?]

SOLUTION We want to find $t$ so that

$$10^6 = 100 \times 1.05^t.$$ 

Thus $1.05^t = 10^4$. Take logarithms of both sides, getting $t \log 1.05 = 4$. Thus $t = \frac{4}{\log 1.05} \approx 188.8$. Because interest is compounded only once per year, we round up to the next year, concluding that it will take 189 years to turn a hundred dollars into a million dollars at 5% annual interest compounded once per year. Hence the comic is correct in stating that there will be at least a million dollars after 190 years. In fact, the comic could have used 189 years instead of 190 years.
LEARNING OBJECTIVES

By the end of this section you should be able to

■ approximate the area under a curve using rectangles;
■ explain the definition of $e$;
■ explain the definition of the natural logarithm and its connection with area;
■ use with the exponential and natural logarithm functions.

3.5 e and the Natural Logarithm

Estimating Area Using Rectangles

The basic idea for calculating the area of a region bounded by a curve is to approximate the region by rectangles. We will illustrate this idea using the curve $y = \frac{1}{x}$ because this curve leads us to $e$, one of the most useful numbers in mathematics, and to the natural logarithm.

We begin by considering the yellow region shown here, whose area is denoted by $\text{area}\left(\frac{1}{x}, 1, 2\right)$. In other words, $\text{area}\left(\frac{1}{x}, 1, 2\right)$ equals the area of the region in the $xy$-plane under the curve $y = \frac{1}{x}$, above the $x$-axis, and between the lines $x = 1$ and $x = 2$.

The next example shows how to obtain a rough estimate of the area in the crudest possible fashion by using only one rectangle.

EXAMPLE 1

Show that

$$\text{area}\left(\frac{1}{x}, 1, 2\right) < 1$$

by enclosing the yellow region above in a single rectangle.

**SOLUTION**

The smallest rectangle (with sides parallel to the coordinate axes) that contains the yellow region is the 1-by-1 square shown here.

Because the yellow region lies inside the 1-by-1 square, the figure here allows us to conclude that the area of the yellow region is less than 1. In other words,

$$\text{area}\left(\frac{1}{x}, 1, 2\right) < 1.$$

Now consider the yellow region shown below:

$$\text{The area of this yellow region is denoted by } \text{area}\left(\frac{1}{x}, 1, 3\right).$$

The area of the yellow region above is denoted by $\text{area}\left(\frac{1}{x}, 1, 3\right)$. In other words, $\text{area}\left(\frac{1}{x}, 1, 3\right)$ equals the area of the region in the $xy$-plane under the curve $y = \frac{1}{x}$, above the $x$-axis, and between the lines $x = 1$ and $x = 3$. 
The next example illustrates the procedure for approximating the area of a region by placing rectangles inside the region.

Show that \( \text{area} \left( \frac{1}{3}, 1, 3 \right) > 1 \) by placing eight rectangles, each with the same size base, inside the yellow region above.

**SOLUTION** Place eight rectangles under the curve, as shown in the figure below:

![Graph of \( y = \frac{1}{x} \)](https://example.com/graph.png)

We have divided the interval \([1, 3]\) into eight intervals of equal size. The interval \([1, 3]\) has length 2. Thus the base of each rectangle has length \( \frac{2}{8} \), which equals \( \frac{1}{4} \).

The base of the first rectangle is the interval \([1, \frac{5}{4}]\). The figure above shows that the height of this first rectangle is \( 1/\frac{5}{4} \), which equals \( \frac{4}{5} \). Because the first rectangle has base \( \frac{1}{4} \) and height \( \frac{4}{5} \), the area of the first rectangle equals \( \frac{1}{4} \cdot \frac{4}{5} \), which equals \( \frac{1}{5} \).

The base of the second rectangle is the interval \([\frac{5}{4}, \frac{3}{2}]\). The height of the second rectangle is \( 1/\frac{3}{2} \), which equals \( \frac{2}{3} \). Thus the area of the second rectangle equals \( \frac{1}{4} \cdot \frac{2}{3} \), which equals \( \frac{1}{6} \).

The area of the third rectangle is computed in the same fashion. Specifically, the third rectangle has base \( \frac{1}{4} \) and height \( 1/\frac{7}{4} \), which equals \( \frac{4}{7} \). Thus the area of the third rectangle equals \( \frac{1}{4} \cdot \frac{4}{7} \), which equals \( \frac{1}{7} \).

The first three rectangles have area \( \frac{1}{5} \), \( \frac{1}{6} \), and \( \frac{1}{7} \), as we have now computed. From this data, you might guess that the eight rectangles have area \( \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \) and \( \frac{1}{12} \). This guess is correct, as you should verify using the same procedure as used above.

Thus the sum of the areas of all eight rectangles is

\[
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} ,
\]

which equals \( \frac{28271}{27720} \). Because these eight rectangles lie inside the yellow region, the area of the region is larger than the sum of the areas of the rectangles. Hence

\[
\text{area} \left( \frac{1}{3}, 1, 3 \right) > \frac{28271}{27720}.
\]

The fraction on the right has a larger numerator than denominator; thus this fraction is larger than 1. Hence without further computation the inequality above shows that

\[
\text{area} \left( \frac{1}{3}, 1, 3 \right) > 1.
\]
In the example above, \( \frac{28271}{27720} \) gives us an estimate for area \( \left( \frac{1}{x}, 1, 3 \right) \). If we want a more accurate estimate, we could use more and thinner rectangles under the curve.

The table below shows the sum of the areas of the rectangles under the curve for several different choices of the number of rectangles. Here we are assuming that the rectangles all have bases of the same size, as in the example above. The sums have been rounded off to five digits.

<table>
<thead>
<tr>
<th>number of rectangles</th>
<th>sum of area of rectangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.0349</td>
</tr>
<tr>
<td>100</td>
<td>1.0920</td>
</tr>
<tr>
<td>1000</td>
<td>1.0978</td>
</tr>
<tr>
<td>10000</td>
<td>1.0985</td>
</tr>
<tr>
<td>100000</td>
<td>1.0986</td>
</tr>
</tbody>
</table>

*Estimates of area \( \left( \frac{1}{x}, 1, 3 \right) \).*

The actual value of area \( \left( \frac{1}{x}, 1, 3 \right) \) is an irrational number whose first five digits are 1.0986, which agrees with the last entry in the table above.

In summary, we can get an accurate estimate of the area of the yellow region by dividing the interval \([1, 3]\) into many small intervals, then computing the sum of the areas of the corresponding rectangles that lie under the curve.

**Defining e**

The area under portions of the curve \( y = \frac{1}{x} \) has some remarkable properties. To discuss these properties, we introduce the following notation, which we have already used for \( c = 2 \) and \( c = 3 \):

\[
\text{area}(\frac{1}{x}, 1, c)
\]

For \( c > 1 \), let area \( \left( \frac{1}{x}, 1, c \right) \) denote the area of the yellow region below:

In other words, area \( \left( \frac{1}{x}, 1, c \right) \) is the area of the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = 1 \) and \( x = c \).

To get a feeling for how area \( \left( \frac{1}{x}, 1, c \right) \) depends on \( c \), consider the following table:
section 3.5  \( e \) and the Natural Logarithm 281

<table>
<thead>
<tr>
<th>( c )</th>
<th>( \text{area}(\frac{1}{x}, 1, c) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.693147</td>
</tr>
<tr>
<td>3</td>
<td>1.098612</td>
</tr>
<tr>
<td>4</td>
<td>1.386294</td>
</tr>
<tr>
<td>5</td>
<td>1.609438</td>
</tr>
<tr>
<td>6</td>
<td>1.791759</td>
</tr>
<tr>
<td>7</td>
<td>1.945910</td>
</tr>
<tr>
<td>8</td>
<td>2.079442</td>
</tr>
<tr>
<td>9</td>
<td>2.197225</td>
</tr>
</tbody>
</table>

The table above agrees with the inequalities that we derived earlier in this section: \( \text{area}(\frac{1}{x}, 1, 2) < 1 \) and \( \text{area}(\frac{1}{x}, 1, 3) > 1 \).

Before reading the next paragraph, pause for a moment to see if you can discover a relationship between any entries in the table above.

If you look for a relationship between entries in the table above, most likely the first thing you will notice is that

\[
\text{area}(\frac{1}{x}, 1, 4) = 2 \cdot \text{area}(\frac{1}{x}, 1, 2).
\]

To see if any other such relationships lurk in the table, we now add a third column showing the ratio of \( \text{area}(\frac{1}{x}, 1, c) \) to \( \text{area}(\frac{1}{x}, 1, 2) \) and a fourth column showing the ratio of \( \text{area}(\frac{1}{x}, 1, c) \) to \( \text{area}(\frac{1}{x}, 1, 3) \):

<table>
<thead>
<tr>
<th>( c )</th>
<th>( \text{area}(\frac{1}{x}, 1, c) )</th>
<th>( \frac{\text{area}(\frac{1}{x}, 1, c)}{\text{area}(\frac{1}{x}, 1, 2)} )</th>
<th>( \frac{\text{area}(\frac{1}{x}, 1, c)}{\text{area}(\frac{1}{x}, 1, 3)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.693147</td>
<td>1.00000</td>
<td>0.63093</td>
</tr>
<tr>
<td>3</td>
<td>1.098612</td>
<td>1.58496</td>
<td>1.00000</td>
</tr>
<tr>
<td>4</td>
<td>1.386294</td>
<td>2.00000</td>
<td>1.26186</td>
</tr>
<tr>
<td>5</td>
<td>1.609438</td>
<td>2.32193</td>
<td>1.46497</td>
</tr>
<tr>
<td>6</td>
<td>1.791759</td>
<td>2.58496</td>
<td>1.63093</td>
</tr>
<tr>
<td>7</td>
<td>1.945910</td>
<td>2.80735</td>
<td>1.77124</td>
</tr>
<tr>
<td>8</td>
<td>2.079442</td>
<td>3.00000</td>
<td>1.89279</td>
</tr>
<tr>
<td>9</td>
<td>2.197225</td>
<td>3.16993</td>
<td>2.00000</td>
</tr>
</tbody>
</table>

The integer entries in the last two columns stand out. We already noted that \( \text{area}(\frac{1}{x}, 1, 4) = 2 \cdot \text{area}(\frac{1}{x}, 1, 2) \); the table above now shows the nice relationships

\[
\text{area}(\frac{1}{x}, 1, 8) = 3 \cdot \text{area}(\frac{1}{x}, 1, 2) \quad \text{and} \quad \text{area}(\frac{1}{x}, 1, 9) = 2 \cdot \text{area}(\frac{1}{x}, 1, 3).
\]

Because \( 4 = 2^2 \) and \( 8 = 2^3 \) and \( 9 = 3^2 \), write these equations more suggestively as

\[
\text{area}(\frac{1}{x}, 1, 2^2) = 2 \cdot \text{area}(\frac{1}{x}, 1, 2); \\
\text{area}(\frac{1}{x}, 1, 2^3) = 3 \cdot \text{area}(\frac{1}{x}, 1, 2); \\
\text{area}(\frac{1}{x}, 1, 3^2) = 2 \cdot \text{area}(\frac{1}{x}, 1, 3).
\]
The equations above suggest the following remarkable formula:

**An area formula**

\[
\text{area}(\frac{1}{x}, 1, c^t) = t \text{ area}(\frac{1}{x}, 1, c)
\]

for every \( c > 1 \) and every \( t > 0 \).

We already know that the formula above holds in three special cases. The formula above will be derived more generally in the next section. For now, assume the evidence from the table above is sufficiently compelling to accept this formula.

The right side of the equation above would be simplified if \( c \) is such that \( \text{area}(\frac{1}{x}, 1, c) = 1 \). Thus we make the following definition:

**Definition of \( e \)**

\( e \) is the number such that

\[
\text{area}(\frac{1}{x}, 1, e) = 1.
\]

Earlier in this section we showed that \( \text{area}(\frac{1}{x}, 1, 2) \) is less than 1 and that \( \text{area}(\frac{1}{x}, 1, 3) \) is greater than 1. Thus for some number between 2 and 3, the area of the region we are considering must equal 1. That number is called \( e \).

The number \( e \) is given a special name because it is so useful in many parts of mathematics. We will see some uses of \( e \) in the next two sections.

The number \( e \) is irrational. Here is a 40-digit approximation of \( e \):

\[ e \approx 2.718281828459045235360287471352662497757 \]

For many practical purposes, 2.718 is a good approximation of \( e \)—the error is about 0.01%.

The fraction \( \frac{19}{7} \) approximates \( e \) fairly well—the error is about 0.1%. The fraction \( \frac{2721}{1001} \) approximates \( e \) even better—the error is about 0.000004%.

Keep in mind that \( e \) is not equal to 2.718 or \( \frac{19}{7} \) or \( \frac{2721}{1001} \). All of these are useful approximations, but \( e \) is an irrational number that cannot be represented exactly as a decimal number or as a fraction.
Defining the Natural Logarithm

The formula
\[ \text{area}\left(\frac{1}{x}, 1, c^t\right) = t \text{area}\left(\frac{1}{x}, 1, c\right) \]

was introduced above. This formula should remind you of the behavior of logarithms with respect to powers. We will now see that the area under the curve \( y = \frac{1}{x} \) is indeed intimately connected with a logarithm.

In the formula above, set \( c \) equal to \( e \) and use the equation \( \text{area}\left(\frac{1}{x}, 1, e\right) = 1 \) to see that
\[ \text{area}\left(\frac{1}{x}, 1, e^t\right) = t. \]

for every positive number \( t \).

Now consider a number \( c > 1 \). We can write \( c \) as a power of \( e \) in the usual fashion: \( c = e^{\log_e c} \). Thus
\[ \text{area}\left(\frac{1}{x}, 1, c\right) = \text{area}\left(\frac{1}{x}, 1, e^{\log_e c}\right) = \log_e c, \]

where the last equality comes from setting \( t = \log_e c \) in the equation from the previous paragraph.

The logarithm with base \( e \), which appeared above, is so useful that it has a special name and special notation.

**Natural logarithm**

For \( c > 0 \) the natural logarithm of \( c \), denoted \( \ln c \), is defined by
\[ \ln c = \log_e c. \]

With this new notation, the equality \( \text{area}\left(\frac{1}{x}, 1, c\right) = \log_e c \) derived above can be rewritten as follows:

**Natural logarithms as areas**

For \( c > 1 \), the natural logarithm of \( c \) is the area of the region below:

In other words,
\[ \ln c = \text{area}\left(\frac{1}{x}, 1, c\right). \]
Properties of the Exponential Function and $\ln$

The function whose value at a number $x$ equals $e^x$ is so important that it also has a special name.

**The exponential function**

The exponential function is the function $f$ defined by

$$f(x) = e^x$$

for every real number $x$.

In Section 3.1 we defined the exponential function with base $b$ to be the function whose value at $x$ is $b^x$. Thus the exponential function defined above is just the exponential function with base $e$. In other words, if no base is mentioned, then assume the base is $e$.

The graph of the exponential function $e^x$ looks similar to the graphs of the functions exponential functions $2^x$ or $3^x$ or any other exponential function with base $b > 1$. Specifically, $e^x$ grows rapidly as $x$ gets large, and $e^x$ is close to 0 for negative values of $x$ with large absolute value.

The domain of the exponential function is the set of real numbers, and the range of the exponential function is the set of positive numbers. Furthermore, the exponential function is an increasing function, as is every function of the form $b^x$ for $b > 1$.

Powers of $e$ have the same algebraic properties as powers of any number. Thus the identities listed below should already be familiar to you. They are included here as a review of key algebraic properties in the specific case of powers of $e$.

**Properties of powers of $e$**

$$
e^0 = 1$$

$$e^1 = e$$

$$e^x e^y = e^{x+y}$$

$$e^{-x} = \frac{1}{e^x}$$

$$\frac{e^x}{e^y} = e^{x-y}$$

$$(e^x)^y = e^{xy}$$

The natural logarithm of a positive number $x$, denoted $\ln x$, equals $\log_e x$. Thus the graph of the natural logarithm looks similar to the graphs of the functions $\log_2 x$ or $\log x$ or $\log_b x$ for any number $b > 1$. Specifically, $\ln x$ grows slowly as $x$ gets large. Furthermore, if $x$ is a small positive number, then $\ln x$ is a negative number with large absolute value, as shown in the following figure:
The graph of $\ln x$ on the interval $[e^{-2}, e^2]$. The same scale is used on both axes to show the slow growth of $\ln x$ and the rapid descent near 0 toward negative numbers with large absolute value.

The domain of $\ln x$ is the set of positive numbers, and the range of $\ln x$ is the set of real numbers. Furthermore, $\ln x$ is an increasing function because it is the inverse of the increasing function $e^x$.

Because the natural logarithm is the logarithm with base $e$, it has all the properties we saw earlier for logarithms with any base. For review, we summarize the key properties here. In the box below, we assume $x$ and $y$ are positive numbers.

**Properties of the natural logarithm**

\[
\begin{align*}
\ln 1 &= 0 \\
\ln e &= 1 \\
\ln(xy) &= \ln x + \ln y \\
\ln \frac{1}{x} &= -\ln x \\
\ln \frac{x}{y} &= \ln x - \ln y \\
\ln(x^t) &= t \ln x
\end{align*}
\]

The exponential function $e^x$ and the natural logarithm $\ln x$ (which equals $\log_e x$) are the inverse functions for each other, just as the functions $2^x$ and $\log_2 x$ are the inverse functions for each other. Thus the exponential function and the natural logarithm exhibit the same behavior as any two functions that are the inverse functions for each other. For review, we summarize here the key properties connecting the exponential function and the natural logarithm:

**Connections between the exponential function and the natural logarithm**

- $\ln y = x$ means $e^x = y$.
- $\ln(e^x) = x$ for every real number $x$.
- $e^{\ln y} = y$ for every positive number $y$.

Recall that in this book, as in most precalculus books, $\log x$ means $\log_{10} x$. However, the natural logarithm is so important that many mathematicians use $\log x$ to denote the natural logarithm rather than the logarithm with base 10.
EXERCISES

The next two exercises emphasize that \( \ln(x + y) \) does not equal \( \ln x + \ln y \).

1. For \( x = 7 \) and \( y = 13 \), evaluate each of the following:
   (a) \( \ln(x + y) \)
   (b) \( \ln x + \ln y \)

2. For \( x = 0.4 \) and \( y = 3.5 \), evaluate each of the following:
   (a) \( \ln(x + y) \)
   (b) \( \ln x + \ln y \)

The next two exercises emphasize that \( \ln(xy) \) does not equal \( \ln x(\ln y) \).

3. For \( x = 3 \) and \( y = 8 \), evaluate each of the following:
   (a) \( \ln(xy) \)
   (b) \( (\ln x)(\ln y) \)

4. For \( x = 1.1 \) and \( y = 5 \), evaluate each of the following:
   (a) \( \ln(xy) \)
   (b) \( (\ln x)(\ln y) \)

The next two exercises emphasize that \( \ln \frac{x}{y} \) does not equal \( \frac{\ln x}{\ln y} \).

5. For \( x = 12 \) and \( y = 2 \), evaluate each of the following:
   (a) \( \ln \frac{x}{y} \)
   (b) \( \frac{\ln x}{\ln y} \)

6. For \( x = 18 \) and \( y = 0.3 \), evaluate each of the following:
   (a) \( \ln \frac{x}{y} \)
   (b) \( \frac{\ln x}{\ln y} \)

Find a number \( y \) such that \( \ln y = 4 \).

Find a number \( x \) such that \( \ln x = 5 \).

Find a number \( c \) such that \( \ln c = 5 \).

Find a number \( x \) such that \( \ln x = -2 \).

Find a number \( x \) such that \( \ln x = -3 \).

Find a number \( t \) such that \( \ln(2t + 1) = -4 \).

Find a number \( w \) such that \( \ln(3w - 2) = 5 \).

Find all numbers \( y \) such that \( \ln(y^2 + 1) = 3 \).

Find all numbers \( r \) such that \( \ln(2r^2 - 3) = -1 \).

Find a number \( x \) such that \( e^{3x - 1} = 2 \).

Find a number \( y \) such that \( e^{4y - 3} = 5 \).

For Exercises 17–28, find all numbers \( x \) that satisfy the given equation.

17. \( \ln(x + 5) - \ln(x - 1) = 2 \)
18. \( \ln(x + 4) - \ln(x - 2) = 3 \)
19. \( \ln(x + 5) + \ln(x - 1) = 2 \)
20. \( \ln(x + 4) + \ln(x + 2) = 2 \)
21. \( \frac{\ln(12x)}{\ln(5x)} = 2 \)
22. \( \frac{\ln(11x)}{\ln(4x)} = 2 \)
23. \( e^{2x} + e^x = 6 \)
24. \( e^{2x} - 4e^x = 12 \)
25. \( e^x + e^{-x} = 6 \)
26. \( e^x + e^{-x} = 8 \)
27. \( (\ln(3x))\ln x = 4 \)
28. \( (\ln(6x))\ln x = 5 \)
29. Find the number \( c \) such that \( \text{area}(\frac{1}{x}, 1, c) = 2 \).
30. Find the number \( c \) such that \( \text{area}(\frac{1}{x}, 1, c) = 3 \).
31. Find the number \( t \) that makes \( e^{t^2 + 6t} \) as small as possible. [Here \( e^{t^2 + 6t} \) means \( e^{(t^2 + 6t)} \).]
32. Find the number \( t \) that makes \( e^{t^2 + 8t + 3} \) as small as possible.
33. Find a number \( y \) such that \( \frac{1 + \ln y}{2 + \ln y} = 0.9 \).
34. Find a number \( w \) such that \( \frac{4 - \ln w}{3 - 5\ln w} = 3.6 \).

For Exercises 35–38, find a formula for \( (f \circ g)(x) \) assuming that \( f \) and \( g \) are the indicated functions.

35. \( f(x) = \ln x \) and \( g(x) = e^{5x} \)
36. \( f(x) = \ln x \) and \( g(x) = e^{4-7x} \)
37. \( f(x) = e^{2x} \) and \( g(x) = \ln x \)
38. \( f(x) = e^{8-5x} \) and \( g(x) = \ln x \)

For each of the functions \( f \) given in Exercises 39–48:

   (a) Find the domain of \( f \).
   (b) Find the range of \( f \).
   (c) Find a formula for \( f^{-1} \).
   (d) Find the domain of \( f^{-1} \).
   (e) Find the range of \( f^{-1} \).

You can check your solutions to part (c) by verifying that \( f^{-1} \circ f = I \) and \( f \circ f^{-1} = I \). (Recall that \( I \) is the function defined by \( I(x) = x \).)

39. \( f(x) = 2 + \ln x \)
40. \( f(x) = 3 - \ln x \)
41. \( f(x) = 4 - 5\ln x \)
42. \( f(x) = -6 + 7\ln x \)
43. \( f(x) = 3e^{2x} \)
44. \( f(x) = 5e^{9x} \)
45. \( f(x) = 4 + \ln(x - 2) \)
46. \( f(x) = 3 + \ln(x + 5) \)
47. \( f(x) = 5 + 6e^{7x} \)
48. \( f(x) = 4 - 2e^{8x} \)

49. What is the area of the region under the curve \( y = \frac{1}{x} \), above the x-axis, and between the lines \( x = 1 \) and \( x = e^2 \)?
50. What is the area of the region under the curve \( y = \frac{1}{x} \), above the x-axis, and between the lines \( x = 1 \) and \( x = e^2 \)?
51 Verify that the last five rectangles in the figure in Example 2 have area \( \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \) and \( \frac{1}{12} \).

52 Consider this figure:

The region under the curve \( y = \frac{1}{x} \), above the x-axis, and between the lines \( x = 1 \) and \( x = 2.5 \).

(a) Calculate the sum of the areas of all six rectangles shown in the figure above.

(b) Explain why the calculation you did in part (a) shows that

\[
\text{area}(\frac{1}{x}, 1, 2.5) < 1.
\]

(c) Explain why the inequality above shows that \( e > 2.5 \).

The following notation is used in Problems 53–56: \( \text{area}(x^2, 1, 2) \) is the area of the region under the curve \( y = x^2 \), above the x-axis, and between the lines \( x = 1 \) and \( x = 2 \), as shown below.

53 Using one rectangle, show that

\( 1 < \text{area}(x^2, 1, 2) \).

54 Using one rectangle, show that

\( \text{area}(x^2, 1, 2) < 4 \).

55 Using four rectangles, show that

\( 1.96 < \text{area}(x^2, 1, 2) \).

56 Using four rectangles, show that

\[
\text{area}(x^2, 1, 2) < 2.72.
\]

[The two problems above show that area\((x^2, 1, 2)\) is in the interval \([1.96, 2.72]\). If we use the midpoint of that interval as an estimate, we get

\[
\text{area}(x^2, 1, 2) \approx \frac{1.96 + 2.72}{2} = 2.34.
\]

This is a very good estimate—the exact value of \( \text{area}(x^2, 1, 2) \) is \( \frac{7}{5} \), which is approximately 2.33.]

57 Explain why

\[
\ln x \approx 2.302585 \log x
\]

for every positive number \( x \).

58 Explain why the solution to part (b) of Exercise 5 in this section is the same as the solution to part (b) of Exercise 5 in Section 3.3.

59 Suppose \( c \) is a number such that \( \text{area}(\frac{1}{x}, 1, c) > 1000 \). Explain why \( c > 2^{1000} \).

The functions \( \cosh \) and \( \sinh \) are defined by

\[
\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}
\]

for every real number \( x \). These functions are called the hyperbolic cosine and hyperbolic sine; they are useful in engineering.

60 Show that \( \cosh \) is an even function.

61 Show that \( \sinh \) is an odd function.

62 Show that

\[
(cosh x)^2 - (sinh x)^2 = 1
\]

for every real number \( x \).

63 Show that \( \cosh x \geq 1 \) for every real number \( x \).

64 Show that

\[
\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y
\]

for all real numbers \( x \) and \( y \).

65 Show that

\[
\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y
\]

for all real numbers \( x \) and \( y \).

66 Show that

\[
(cosh x + sinh x)^t = cosh(t x) + sinh(t x)
\]

for all real numbers \( x \) and \( t \).

67 Show that if \( x \) is very large, then

\[
cosh x \approx \sinh x = \frac{e^x}{2}.
\]
68 Show that the range of sinh is the set of real numbers.

69 Show that sinh is a one-to-one function and that its inverse is given by the formula

$$\sinh^{-1}(y) = \ln(y + \sqrt{y^2 + 1})$$

for every real number $y$.

70 Show that the range of cosh is the interval $[1, \infty)$.

71 Suppose $f$ is the function defined by

$$f(x) = \cosh x$$

for every $x \geq 0$. In other words, $f$ is defined by the same formula as cosh, but the domain of $f$ is the interval $[0, \infty)$ and the domain of cosh is the set of real numbers. Show that $f$ is a one-to-one function and that its inverse is given by the formula

$$f^{-1}(y) = \ln(y + \sqrt{y^2 - 1})$$

for every $y \geq 1$.

72 Write a description of how the shape of the St. Louis Gateway Arch is related to the graph of $\cosh x$. You should be able to find the necessary information using an appropriate web search.

**WORKED-OUT SOLUTIONS to Odd-Numbered Exercises**

The next two exercises emphasize that $\ln(x + y)$ does not equal $\ln x + \ln y$.

1 For $x = 7$ and $y = 13$, evaluate each of the following:

(a) $\ln(x + y)$

(b) $\ln x + \ln y$

**SOLUTION**

(a) $\ln(7 + 13) = \ln 20 \approx 2.99573$

(b) $\ln 7 + \ln 13 \approx 1.94591 + 2.56495$

$\approx 4.51086$

The next two exercises emphasize that $\ln(x \cdot y)$ does not equal $(\ln x)(\ln y)$.

3 For $x = 3$ and $y = 8$, evaluate each of the following:

(a) $\ln(x \cdot y)$

(b) $(\ln x)(\ln y)$

**SOLUTION**

(a) $\ln(3 \cdot 8) = \ln 24 \approx 3.17805$

(b) $(\ln 3)(\ln 8) \approx (1.09861)(2.07944)$

$\approx 2.2845$

The next two exercises emphasize that $\ln \frac{x}{y}$ does not equal $\frac{\ln x}{\ln y}$.

5 For $x = 12$ and $y = 2$, evaluate each of the following:

(a) $\ln \frac{x}{y}$

(b) $\frac{\ln x}{\ln y}$

**SOLUTION**

(a) $\ln \frac{12}{2} = \ln 6 \approx 1.79176$

(b) $\frac{\ln 12}{\ln 2} \approx \frac{2.48490665}{0.6931472} \approx 3.58496$

7 Find a number $y$ such that $\ln y = 4$.

**SOLUTION** Recall that $\ln y$ is simply shorthand for $\log_e y$. Thus the equation $\ln y = 4$ can be rewritten as $\log_e y = 4$. The definition of a logarithm now implies that $y = e^4$.

9 Find a number $x$ such that $\ln x = -2$.

**SOLUTION** Recall that $\ln x$ is simply shorthand for $\log_e x$. Thus the equation $\ln x = -2$ can be rewritten as $\log_e x = -2$. The definition of a logarithm now implies that $x = e^{-2}$.

11 Find a number $t$ such that $\ln(2t + 1) = -4$.

**SOLUTION** The equation $\ln(2t + 1) = -4$ implies that $e^{-4} = 2t + 1$.

Solving this equation for $t$, we get

$$t = \frac{e^{-4} - 1}{2}.$$
13 Find all numbers $y$ such that $\ln(y^2 + 1) = 3$.

**SOLUTION** The equation $\ln(y^2 + 1) = 3$ implies that
e^{3} = y^2 + 1.
Thus $y^2 = e^3 - 1$, which means that $y = \sqrt{e^3 - 1}$ or $y = -\sqrt{e^3 - 1}$.

15 Find a number $x$ such that $e^{3x-1} = 2$.

**SOLUTION** The equation $e^{3x-1} = 2$ implies that
$3x - 1 = \ln 2$.
Solving this equation for $x$, we get
$x = \frac{1 + \ln 2}{3}$.

For Exercises 17–28, find all numbers $x$ that satisfy the given equation.

17 $\ln(x + 5) - \ln(x - 1) = 2$

**SOLUTION** Our equation can be rewritten as follows:
$$2 = \ln(x + 5) - \ln(x - 1)$$
$$= \ln \frac{x + 5}{x - 1}.$$ 
Thus $\frac{x + 5}{x - 1} = e^2$.
We can solve the equation above for $x$, getting
$$x = \frac{e^2 + 5}{e^2 - 1}.$$

19 $\ln(x + 5) + \ln(x - 1) = 2$

**SOLUTION** Our equation can be rewritten as follows:
$$2 = \ln(x + 5) + \ln(x - 1)$$
$$= \ln((x + 5)(x - 1))$$
$$= \ln(x^2 + 4x - 5).$$
Thus $x^2 + 4x - 5 = e^2$,
which implies that
$$x^2 + 4x - (e^2 + 5) = 0.$$
We can solve the equation above using the quadratic formula, getting $x = -2 + \sqrt{9 + e^2}$ or $x = -2 - \sqrt{9 + e^2}$.
However, both $x + 5$ and $x - 1$ are negative if $x = -2 - \sqrt{9 + e^2}$; because the logarithm of a negative number is undefined, we must discard this root of the equation above. We conclude that the only value of $x$ satisfying the equation $\ln(x + 5) + \ln(x - 1) = 2$ is $x = -2 + \sqrt{9 + e^2}$.

21 $\frac{\ln(12x)}{\ln(5x)} = 2$

**SOLUTION** Our equation can be rewritten as follows:
$$2 = \frac{\ln(12x)}{\ln(5x)}$$
$$= \frac{\ln 12 + \ln x}{\ln 5 + \ln x}.$$ 
Solving this equation for $\ln x$ (the first step in doing this is to multiply both sides by the denominator $\ln 5 + \ln x$), we get
$$\ln x = \ln 12 - 2 \ln 5$$
$$= \ln 12 - \ln 25$$
$$= \ln \frac{12}{25}.$$ 
Thus $x = \frac{12}{25}$.

23 $e^{2x} + e^x = 6$

**SOLUTION** Note that $e^{2x} = (e^x)^2$. This suggests that we let $t = e^x$. Then the equation above can be rewritten as
t^2 + t - 6 = 0.
The solutions to this equation (which can be found either by using the quadratic formula or by factoring) are $t = -3$ and $t = 2$. Thus $e^x = -3$ or $e^x = 2$. However, there is no real number $x$ such that $e^x = -3$ (because $e^x$ is positive for every real number $x$), and thus we must have $e^x = 2$. Thus $x = \ln 2 \approx 0.693147$.

25 $e^x + e^{-x} = 6$

**SOLUTION** Let $t = e^x$. Then the equation above can be rewritten as
t + \frac{1}{t} - 6 = 0.
Multiply both sides by $t$, giving the equation
t^2 - 6t + 1 = 0.
The solutions to this equation (which can be found by using the quadratic formula) are $t = 3 - 2\sqrt{2}$ and $t = 3 + 2\sqrt{2}$. Thus $e^x = 3 - 2\sqrt{2}$ or $e^x = 3 + 2\sqrt{2}$. Thus the solutions to the original equation are $x = \ln(3 - 2\sqrt{2})$ and $x = \ln(3 + 2\sqrt{2})$.

27 $(\ln(3x)) \ln x = 4$

**SOLUTION** Our equation can be rewritten as follows:
$$4 = (\ln(3x)) \ln x$$
$$= (\ln x + \ln 3) \ln x$$
$$= (\ln x)^2 + (\ln 3)(\ln x).$$
Letting \( y = \ln x \), we can rewrite the equation above as
\[
y^2 + (\ln 3) y - 4 = 0.
\]
Use the quadratic formula to solve the equation above for \( y \), getting
\[
y \approx -2.62337 \quad \text{or} \quad y \approx 1.52476.
\]
Thus
\[
\ln x \approx -2.62337 \quad \text{or} \quad \ln x \approx 1.52476,
\]
which means that
\[
x \approx e^{-2.62337} \approx 0.072558
\]
or
\[
x \approx e^{1.52476} \approx 4.59403.
\]
29 Find the number \( c \) such that area(\( \frac{1}{x}, 1, c \)) = 2.

**SOLUTION** Because \( 2 = \text{area}(\frac{1}{x}, 1, c) = \ln c \), we see that \( c = e^2 \).

31 Find the number \( t \) that makes \( e^{t^2 + 6t} \) as small as possible.

**SOLUTION** Because \( e^x \) is an increasing function of \( x \), the number \( e^{t^2 + 6t} \) will be as small as possible when \( t^2 + 6t \) is as small as possible. To find when \( t^2 + 6t \) is as small as possible, we complete the square:
\[
t^2 + 6t = (t + 3)^2 - 9.
\]
The equation above shows that \( t^2 + 6t \) is as small as possible when \( t = -3 \).

33 Find a number \( y \) such that
\[
\frac{1 + \ln y}{2 + \ln y} = 0.9.
\]

**SOLUTION** Multiplying both sides of the equation above by \( 2 + \ln y \) and then solving for \( \ln y \) gives \( \ln y = 8 \). Thus \( y = e^8 \approx 2980.96 \).

**For Exercises 35–38, find a formula for \((f \circ g)(x)\) assuming that \( f \) and \( g \) are the indicated functions.**

35 \( f(x) = \ln x \) and \( g(x) = e^{5x} \)

**SOLUTION**
\[
(f \circ g)(x) = f(g(x)) = f(e^{5x}) = \ln(e^{5x}) = 5x
\]

37 \( f(x) = e^{2x} \) and \( g(x) = \ln x \)

**SOLUTION**
\[
(f \circ g)(x) = f(g(x)) = f(\ln x) = e^{2\ln x} = (e^{\ln x})^2 = x^2
\]

**For each of the functions \( f \) given in Exercises 39–48:**

(a) Find the domain of \( f \).
(b) Find the range of \( f \).
(c) Find a formula for \( f^{-1} \).
(d) Find the domain of \( f^{-1} \).
(e) Find the range of \( f^{-1} \).

You can check your solutions to part (c) by verifying that \( f^{-1} \circ f = I \) and \( f \circ f^{-1} = I \). (Recall that \( I \) is the function defined by \( I(x) = x \).)

39 \( f(x) = 2 + \ln x \)

**SOLUTION**
(a) The expression \( 2 + \ln x \) makes sense for all positive numbers \( x \). Thus the domain of \( f \) is the set of positive numbers.
(b) The range of \( f \) is obtained by adding 2 to each number in the range of \( \ln x \). Because the range of \( \ln x \) is the set of real numbers, the range of \( f \) is also the set of real numbers.
(c) The expression above shows that \( f^{-1} \) is given by the expression
\[
f^{-1}(y) = e^{y-2}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set of real numbers.
(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set of positive numbers.

41 \( f(x) = 4 - 5 \ln x \)

**SOLUTION**
(a) The expression \( 4 - 5 \ln x \) makes sense for all positive numbers \( x \). Thus the domain of \( f \) is the set of positive numbers.
(b) The range of \( f \) is obtained by multiplying each number in the range of \( \ln x \) by \(-5\) and then adding 4. Because the range of \( \ln x \) is the set of real numbers, the range of \( f \) is also the set of real numbers.
(c) The expression above shows that \( f^{-1} \) is given by the expression
\[
f^{-1}(y) = e^{(4-y)/5}.
\]
(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set of real numbers.
(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set of positive numbers.
43  \( f(x) = 3e^{2x} \)

**SOLUTION**

(a) The expression \( 3e^{2x} \) makes sense for all real numbers \( x \). Thus the domain of \( f \) is the set of real numbers.

(b) To find the range of \( f \), we need to find the numbers \( y \) such that
\[
y = 3e^{2x}
\]
for some \( x \) in the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a real number \( x \). To solve this equation for \( x \), divide both sides by 3, getting \( \frac{y}{3} = e^{2x} \), which implies that \( 2x = \ln \frac{y}{3} \). Thus
\[
x = \frac{\ln \frac{y}{3}}{2}.
\]
The expression above on the right makes sense for every positive number \( y \) and produces a real number \( x \). Thus the range of \( f \) is the set of positive numbers.

(c) The expression above shows that \( f^{-1} \) is given by the expression
\[
f^{-1}(y) = \frac{\ln \frac{y}{3}}{2}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set of positive numbers.

(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the interval \((2, \infty)\).

47  \( f(x) = 5 + 6e^{7x} \)

**SOLUTION**

(a) The expression \( 5 + 6e^{7x} \) makes sense for all real numbers \( x \). Thus the domain of \( f \) is the set of real numbers.

(b) To find the range of \( f \), we need to find the numbers \( y \) such that
\[
y = 5 + 6e^{7x}
\]
for some \( x \) in the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a real number \( x \). To solve this equation for \( x \), subtract 5 from both sides, then divide both sides by 6, getting \( \frac{y-5}{6} = e^{7x} \), which implies that \( 7x = \ln \frac{y-5}{6} \). Thus
\[
x = \frac{\ln \frac{y-5}{6}}{7}.
\]
The expression above on the right makes sense for every \( y > 5 \) and produces a real number \( x \). Thus the range of \( f \) is the interval \((5, \infty)\).

(c) The expression above shows that \( f^{-1} \) is given by the expression
\[
f^{-1}(y) = \frac{\ln \frac{y-5}{6}}{7}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the set of real numbers.

(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set of real numbers.

45  \( f(x) = 4 + \ln(x - 2) \)

**SOLUTION**

(a) The expression \( 4 + \ln(x - 2) \) makes sense when \( x > 2 \). Thus the domain of \( f \) is the interval \((2, \infty)\).

(b) To find the range of \( f \), we need to find the numbers \( y \) such that
\[
y = 4 + \ln(x - 2)
\]
for some \( x \) in the domain of \( f \). In other words, we need to find the values of \( y \) such that the equation above can be solved for a number \( x > 2 \). To solve this equation for \( x \), subtract 4 from both sides, getting \( y - 4 = \ln(x - 2) \), which implies that \( x - 2 = e^{y-4} \). Thus
\[
x = 2 + e^{y-4}.
\]
The expression above on the right makes sense for every real number \( y \) and produces a number \( x > 2 \) (because \( e \) raised to any power is positive). Thus the range of \( f \) is the set of real numbers.

(c) The expression above shows that \( f^{-1} \) is given by the expression
\[
f^{-1}(y) = \frac{\ln \frac{y-5}{6}}{7}.
\]

(d) The domain of \( f^{-1} \) equals the range of \( f \). Thus the domain of \( f^{-1} \) is the interval \((5, \infty)\).

(e) The range of \( f^{-1} \) equals the domain of \( f \). Thus the range of \( f^{-1} \) is the set of real numbers.

49 What is the area of the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = 1 \) and \( x = e^2 \)?

**SOLUTION** The area of this region is \( \ln(e^2) \), which equals 2.
3.6 Approximations and area with \( e \) and \( \ln \)

**LEARNING OBJECTIVES**

By the end of this section you should be able to

- approximate \( \ln(1 + t) \) for small values of \( |t| \);
- approximate \( e^t \) for small values of \( |t| \);
- approximate \( (1 + \frac{r}{x})^x \) when \( |x| \) is much larger than \( |r| \);
- explain the area formula that led to \( e \) and the natural logarithm.

**Approximation of the Natural Logarithm**

The next example leads to an important result.

**EXAMPLE 1**

Discuss the behavior of \( \ln(1 + t) \) for \( |t| \) a small number.

**SOLUTION**

The table shows the value of \( \ln(1 + t) \), rounded off to six significant digits, for some small values of \( |t| \). This table leads us to guess that \( \ln(1 + t) \approx t \) if \( |t| \) is a small number, with the approximation becoming more accurate as \( |t| \) becomes smaller.

The graph here confirms that

\[ \ln(1 + t) \approx t \]

if \( |t| \) is small. At this scale, we cannot see the difference between \( \ln(1 + t) \) and \( t \) for \( t \) in the interval \([-0.05, 0.05]\).

To explain the behavior in the example above, suppose \( t > 0 \). Recall from the previous section that \( \ln(1 + t) \) is the area of the yellow region below. If \( t \) is a small positive number, then the area of this region is approximately the area of the rectangle below. This rectangle has base \( t \) and height 1; thus the rectangle has area \( t \). We conclude that \( \ln(1 + t) \approx t \).
The result below demonstrates again why the natural logarithm deserves the name *natural*. No base for logarithms other than $e$ produces such a nice approximation formula.

**Approximation of the natural logarithm**

If $|t|$ is small, then $\ln(1 + t) \approx t$.

Consider now the figure below, where we assume $t$ is positive but not necessarily small. In this figure, $\ln(1 + t)$ equals the area of the yellow region.

The area of the lower rectangle is less than the area of the yellow region. The area of the yellow region is less than the area of the large rectangle.

The yellow region above contains the lower rectangle; thus the lower rectangle has a smaller area. The lower rectangle has base $t$ and height $\frac{1}{1+t}$ and hence has area $\frac{t}{1+t}$. Thus

$$\frac{t}{1+t} < \ln(1 + t).$$

The large rectangle in the figure above has base $t$ and height 1 and thus has area $t$. The yellow region above is contained in the large rectangle; thus the large rectangle has a bigger area. In other words,

$$\ln(1 + t) < t.$$

Putting together the inequalities from the previous two paragraphs, we have the result below.

**Inequalities with the natural logarithm**

If $t > 0$, then \( \frac{t}{1 + t} < \ln(1 + t) < t \).

This result is valid for all positive numbers $t$, regardless of whether $t$ is small or large.
Approximations with the Exponential Function

Now we turn to approximations of $e^x$. In the next section, we will see important applications of these approximations involving $e$.

**Example 2**

Discuss the behavior of $e^x$ for $|x|$ a small number.

**Solution**

The table shows the value of $e^x$, rounded off appropriately, for some small values of $|x|$. This table leads us to guess that $e^x \approx 1 + x$ if $|x|$ is a small number, with the approximation becoming more accurate as $|x|$ becomes smaller.

The graph here confirms that $e^x \approx 1 + x$ if $|x|$ is small. At this scale, we cannot see the difference between $e^x$ and $1 + x$ for $x$ in the interval $[-0.05, 0.05]$.

To explain the behavior in the example above, suppose $|x|$ is small. Then, as we already know, $x \approx \ln(1 + x)$. Thus

$$e^x \approx e^{\ln(1+x)} = 1 + x.$$ 

Hence we have the following result:

**Approximation of the exponential function**

If $|x|$ is small, then

$$e^x \approx 1 + x.$$ 

Another useful approximation gives good estimates for $e^r$ even when $r$ is not small. As an example, consider the following table of values of $(1 + \frac{1}{x})^x$ for large values of $x$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$(1 + \frac{1}{x})^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2.70481</td>
</tr>
<tr>
<td>1000</td>
<td>2.71692</td>
</tr>
<tr>
<td>10000</td>
<td>2.71815</td>
</tr>
<tr>
<td>100000</td>
<td>2.71827</td>
</tr>
<tr>
<td>1000000</td>
<td>2.71828</td>
</tr>
</tbody>
</table>

Values of $(1 + \frac{1}{x})^x$, rounded off to six digits.

You may recognize the last entry in the table above as the value of $e$, rounded off to six digits. In other words, it appears that $(1 + \frac{1}{x})^x \approx e$ for large values of $x$. We will now see that an even more general approximation is valid.
Let \( r \) be any number, and suppose \( x \) is a number with \( |x| \) much larger than \( |r| \). Thus \( \left| \frac{r}{x} \right| \) is small. Then, as we already know, \( e^{r/x} \approx 1 + \frac{r}{x} \). Thus

\[
e^r = (e^{r/x})^x \approx (1 + \frac{r}{x})^x.
\]

Hence we have the following result:

**Approximation of the exponential function**

If \( |x| \) is much larger than \( |r| \), then

\[
(1 + \frac{r}{x})^x \approx e^r.
\]

For example, taking \( r = 1 \), this approximation shows that

\[
(1 + \frac{1}{x})^x \approx e
\]

for large values of \( x \), confirming the results indicated by the table above.

---

**EXAMPLE 3**

Estimate \( 1.00002^{40} \).

**SOLUTION** We have

\[
1.00002^{40} = (1 + 0.00002)^{40} = (1 + \frac{0.00002}{40})^{40} = (1 + \frac{0.00002}{40})^{40} \\
\approx e^{0.0008} \\
\approx 1 + 0.0008 \\
= 1.0008,
\]

where the first approximation comes from the box above (40 is indeed much larger than 0.0008) and the second approximation comes from the box before that (0.0008 is indeed small).

This estimate is quite accurate—the first eight digits in the exact value of \( 1.00002^{40} \) are 1.0008003.

The same reasoning as used in the example above leads to the following result.

**Raising 1 plus a small number to a power**

Suppose \( t \) and \( n \) are numbers such that \( |t| \) and \( |nt| \) are small. Then

\[
(1 + t)^n \approx 1 + nt.
\]

The estimate in the box above holds because

\[
(1 + t)^n = \left(1 + \frac{nt}{n}\right)^n \approx e^{nt} \approx 1 + nt.
\]
An Area Formula

The area formula

\[ \text{area} \left( \frac{1}{x}, 1, c^t \right) = t \text{area} \left( \frac{1}{x}, 1, c \right) \]

played a crucial role in the previous section, leading to the definitions of \( e \) and the natural logarithm. Now we explain why this formula is true.

We start by introducing some slightly more general notation.

\[ \text{area} \left( \frac{1}{x}, b, c \right) \]

For positive numbers \( b \) and \( c \) with \( b < c \), let \( \text{area} \left( \frac{1}{x}, b, c \right) \) denote the area of the yellow region below:

In other words, \( \text{area} \left( \frac{1}{x}, b, c \right) \) is the area of the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = b \) and \( x = c \).

The solution to the next example contains the key idea that will help us derive the area formula. In this example and the other results in the remainder of this section, we cannot use the equation \( \text{area} \left( \frac{1}{x}, 1, c \right) = \ln c \). Using that equation would be circular reasoning because we are now trying to show that \( \text{area} \left( \frac{1}{x}, 1, c^t \right) = t \text{area} \left( \frac{1}{x}, 1, c \right) \), which was used to show that \( \text{area} \left( \frac{1}{x}, 1, c \right) = \ln c \).

**EXAMPLE 4**

Explain why \( \text{area} \left( \frac{1}{x}, 1, 2 \right) = \text{area} \left( \frac{1}{x}, 2, 4 \right) = \text{area} \left( \frac{1}{x}, 4, 8 \right) \).

**SOLUTION**

We need to explain why the three regions below have the same area.

Stretching the region on the left horizontally by a factor of 2 and vertically by a factor of \( \frac{1}{2} \) gives the center region. Thus these two regions have the same area.

Define a function \( f \) with domain \([1, 2]\) by

\[ f(x) = \frac{1}{x} \]

and define a function \( g \) with domain \([2, 4]\) by

\[ g(x) = \frac{1}{2} f \left( \frac{x}{2} \right) = \frac{1}{2} \cdot \frac{1}{x} = \frac{1}{x}. \]
Our results on function transformations (see Section 1.3) show that the graph of \( g \) is obtained from the graph of \( f \) by stretching horizontally by a factor of 2 and stretching vertically by a factor of \( \frac{1}{2} \). In other words, the region above in the center is obtained from the region above on the left by stretching horizontally by a factor of 2 and stretching vertically by a factor of \( \frac{1}{2} \). The Area Stretch Theorem (see Appendix A) now implies that the area of the region in the center is \( 2 \cdot \frac{1}{2} \) times the area of the region on the left. Because \( 2 \cdot \frac{1}{2} = 1 \), this implies that the two regions have the same area.

To show that the region above on the right has the same area as the region above on the left, follow the same procedure, but now define a function \( h \) with domain \([4, 8]\) by

\[
h(x) = \frac{1}{4} f\left(\frac{x}{4}\right) = \frac{1}{4} \frac{1}{x} = \frac{1}{x}.
\]

The graph of \( h \) is obtained from the graph of \( f \) by stretching horizontally by a factor of 4 and stretching vertically by a factor of \( \frac{1}{4} \). The Area Stretch Theorem now implies that the region on the right has the same area as the region on the left.

By inspecting a table of numbers in the previous section, we noticed that

\[
\text{area}\left(\frac{1}{x}, 1, 2^3\right) = 3 \text{area}\left(\frac{1}{x}, 1, 2\right).
\]

The next result shows why this is true.

---

**EXAMPLE 5**

**SOLUTION**

Because \( 2^3 = 8 \), partition the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = 1 \) and \( x = 8 \) into three regions, as shown here.

The previous example shows that each of these three regions has the same area. Thus \( \text{area}\left(\frac{1}{x}, 1, 2^3\right) = 3 \text{area}\left(\frac{1}{x}, 1, 2\right) \).

In the example above, there is nothing special about the number 2. We can replace 2 by any number \( c > 1 \), and using the same reasoning as in the two previous examples we can conclude that

\[
\text{area}\left(\frac{1}{x}, 1, c^3\right) = 3 \text{area}\left(\frac{1}{x}, 1, c\right).
\]

Furthermore, there is nothing special about the number 3 in the equation above. Replace 3 with a positive integer \( t \) and use the same reasoning as above to show that

\[
\text{area}\left(\frac{1}{x}, 1, c^t\right) = t \text{area}\left(\frac{1}{x}, 1, c\right).
\]

At this point, we have derived the desired area formula with the restriction that \( t \) must be a positive integer. If you have understood everything up to this point, this is an excellent achievement and a reasonable stopping place. If you want to understand the full area formula, then work through the following example, which removes the restriction that \( t \) be an integer.

*If you learn calculus, you will encounter many of the ideas we have been discussing concerning area. The part of calculus called integral calculus focuses on area.*
EXAMPLE 6

Explain why

\[ \text{area}\left(\frac{1}{x}, 1, c^t\right) = t \text{area}\left(\frac{1}{x}, 1, c\right) \]

for every \( c > 1 \) and every \( t > 0 \).

SOLUTION  First we will verify the desired equation when \( t \) is a positive rational number. So suppose \( t = \frac{m}{n} \), where \( m \) and \( n \) are positive integers. Using the restricted area formula that we have already derived, but replacing \( c \) by \( c^{n/m} \) and replacing \( t \) by \( m \), we have

\[ \text{area}\left(\frac{1}{x}, 1, (c^{n/m})^m\right) = m \text{area}\left(\frac{1}{x}, 1, c^{n/m}\right). \]

Because \( (c^{n/m})^m = c^n \), we can rewrite the equation above as

\[ \text{area}\left(\frac{1}{x}, 1, c^n\right) = m \text{area}\left(\frac{1}{x}, 1, c^{n/m}\right). \]

By the restricted area formula that we have already derived, the left side of the equation above equals \( n \text{area}\left(\frac{1}{x}, 1, c\right) \). Thus

\[ n \text{area}\left(\frac{1}{x}, 1, c\right) = m \text{area}\left(\frac{1}{x}, 1, c^{n/m}\right). \]

Dividing the both sides of equation above by \( m \) and reversing the two sides gives

\[ \text{area}\left(\frac{1}{x}, 1, c^{n/m}\right) = \frac{n}{m} \text{area}\left(\frac{1}{x}, 1, c\right). \]

In other words, we have now shown that

\[ \text{area}\left(\frac{1}{x}, 1, c^t\right) = t \text{area}\left(\frac{1}{x}, 1, c\right) \]

whenever \( t \) is a positive rational number. Because every positive number can be approximated as closely as we like by a positive rational number, this implies that the equation above holds whenever \( t \) is a positive number.

EXERCISES

For Exercises 1–16, estimate the indicated value without using a calculator.

1 \( \ln 1.003 \)  
2 \( \ln 1.0007 \)  
3 \( \ln 0.993 \)  
4 \( \ln 0.9996 \)  
5 \( \ln 3.0012 - \ln 3 \)  
6 \( \ln 4.001 - \ln 4 \)  
7 \( e^{0.0013} \)  
8 \( e^{0.00092} \)  
9 \( e^{-0.0083} \)  
10 \( e^{-0.00046} \)  
11 \( e^{8.997} \)  
12 \( e^{5.984} \)  
13 \( \left(\frac{e^{7.001}}{e^2}\right)^2 \)  
14 \( \left(\frac{e^{8.0002}}{e^8}\right)^3 \)  
15 \( 1.00001^{34.5} \)  
16 \( 1.0002^{7.3} \)

For Exercises 17–22, estimate the given number. Your calculator will be unable to evaluate directly the expressions in these exercises. Thus you will need to do more than button pushing for these exercises.

17 \( (1 + \frac{3}{10})^{100} \)  
18 \( (1 + \frac{5}{10})^{1000} \)  
19 \( (1 - \frac{4}{9})^{100} \)  
20 \( (1 - \frac{2}{8})^{800} \)  
21 \( (1 + 10^{-100})^{2 \cdot 10^{1000}} \)  
22 \( (1 + 10^{-100})^{3 \cdot 10^{1000}} \)
23 Estimate the slope of the line containing the points 
\((5, \ln 5)\) and \((5 + 10^{-100}, \ln(5 + 10^{-100}))\).

24 Estimate the slope of the line containing the points 
\((4, \ln 4)\) and \((4 + 10^{-1000}, \ln(4 + 10^{-1000}))\).

25 Suppose \(t\) is a small positive number. Estimate the slope of the line containing the points 
\((4, e^t)\) and \((4 + t, e^{4+t})\).

26 Suppose \(r\) is a small positive number. Estimate the slope of the line containing the points 
\((7, e^r)\) and \((7 + r, e^{7+r})\).

27 Suppose \(r\) is a small positive number. Estimate the slope of the line containing the points 
\((e^r, 6)\) and \((e^{2+r}, 6 + r)\).

PROBLEMS

33 Show that
\[
\frac{1}{10^{20} + 1} < \ln(1 + 10^{-20}) < \frac{1}{10^{20}}.
\]

34 Estimate the value of
\[10^{50}(\ln(10^{50} + 1) - \ln(10^{50}))\].

35 (a) Using a calculator, verify that
\[
\log(1 + t) \approx 0.434294t
\]
for some small numbers \(t\) (for example, try \(t = 0.001\) and then smaller values of \(t\)).

(b) Explain why the approximation above follows from the approximation \(\ln(1 + t) \approx t\).

36 (a) Using a calculator or computer, verify that
\[
2^t - 1 \approx 0.693147t
\]
for some small numbers \(t\) (for example, try \(t = 0.001\) and then smaller values of \(t\)).

(b) Explain why \(2^t = e^{t \ln 2}\) for every number \(t\).

(c) Explain why the approximation in part (a) follows from the approximation \(e^t \approx 1 + t\).

Part (b) of the problem below gives another reason why the natural logarithm deserves the name “natural”.

37 Suppose \(x\) is a positive number.

(a) Explain why \(x^t = e^{t \ln x}\) for every number \(t\).

(b) Explain why
\[
\frac{x^t - 1}{t} \approx \ln x
\]
if \(t\) is close to 0.

38 (a) Using a calculator or computer, verify that
\[
(1 + \frac{\ln 10}{x})^x \approx 10
\]
for large values of \(x\) (for example, try \(x = 1000\) and then larger values of \(x\)).

(b) Explain why the approximation above follows from the approximation \((1 + \frac{t}{x})^x \approx e^t\).

39 Using a calculator, discover a formula for a good approximation of
\[\ln(2 + t) - \ln 2\]
for small values of \(t\) (for example, try \(t = 0.04, t = 0.02, t = 0.01\), and then smaller values of \(t\)). Then explain why your formula is indeed a good approximation.

40 Show that for every positive number \(c\), we have
\[\ln(c + t) - \ln c \approx \frac{t}{c}\]
for small values of \(t\).

41 Show that for every number \(c\), we have
\[e^{c+t} - e^c \approx te^c\]
for small values of \(t\).

The next two problems combine to show that
\[1 + t < e^t < (1 + t)^{1+t}\]
if \(t > 0\).
42 Show that if \( t > 0 \), then \( 1 + t < e^t \).
43 Show that if \( t > 0 \), then \( e^t < (1 + t)^{1+t} \).

The next two problems combine to show that

\[ (1 + \frac{1}{x})^x < e < (1 + \frac{1}{x})^{x+1} \]

if \( x > 0 \).

44 Show that if \( x > 0 \), then \( (1 + \frac{1}{x})^x < e \).
45 Show that if \( x > 0 \), then \( e < (1 + \frac{1}{x})^{x+1} \).
46 (a) Show that

\[ 1.01^{100} < e < 1.01^{101} \]

(b) Explain why

\[ \frac{1.01^{100} + 1.01^{101}}{2} \]

is a reasonable estimate of \( e \).

47 Show that

\[ \text{area}(\frac{1}{x}, \frac{1}{x}, 1) = \text{area}(\frac{1}{x}, 1, b) \]

for every number \( b > 1 \).

**WORKED-OUT SOLUTIONS to Odd-Numbered Exercises**

For Exercises 1–16, estimate the indicated value without using a calculator.

1 \( \ln 1.003 \)
   **SOLUTION**
   \[ \ln 1.003 = \ln(1 + 0.003) \approx 0.003 \]
2 \( \ln 0.993 \)
   **SOLUTION**
   \[ \ln 0.993 = \ln(1 + (-0.007)) \approx -0.007 \]
3 \( \ln 3.0012 - \ln 3 \)
   **SOLUTION**
   \[ \ln 3.0012 - \ln 3 = \frac{\ln 3.0012}{3} = \ln 1.0004 \]
   \[ = \ln(1 + 0.0004) \approx 0.0004 \]
4 \( e^{0.0013} \)
   **SOLUTION**
   \[ e^{0.0013} \approx 1 + 0.0013 = 1.0013 \]
5 \( e^{-0.0083} \)
   **SOLUTION**
   \[ e^{-0.0083} \approx 1 + (-0.0083) = 0.9917 \]
6 \( \frac{e^9}{e^{8.997}} \)
   **SOLUTION**
   \[ \frac{e^9}{e^{8.997}} = e^{9-8.997} = e^{0.003} \approx 1 + 0.003 = 1.003 \]
7 \( \left( \frac{e^{7.001}}{e^7} \right)^2 \)
   **SOLUTION**
   \[ \left( \frac{e^{7.001}}{e^7} \right)^2 = \left( e^{7.001-7} \right)^2 = \left( e^{0.001} \right)^2 \]
   \[ = e^{0.002} \approx 1 + 0.002 = 1.002 \]
8 \( 1.00001^{34.5} \)
   **SOLUTION**
   Because 0.00001 and 34.5 \( \times \) 0.00001 are both small, we have
   \[ 1.00001^{34.5} \approx 1 + 34.5 \times 0.00001 \]
   \[ = 1.000345. \]
For Exercises 17–22, estimate the given number. Your calculator will be unable to evaluate directly the expressions in these exercises. Thus you will need to do more than button pushing for these exercises.

17 \( \left( 1 + \frac{3}{10^{100}} \right)^{10^{100}} \)

**Solution** \( \left( 1 + \frac{3}{10^{100}} \right)^{10^{100}} \approx e^3 \approx 20.09 \)

19 \( \left( 1 - \frac{4}{980} \right)^{980} \)

**Solution** \( \left( 1 - \frac{4}{980} \right)^{980} \approx e^{-4} \approx 0.01832 \)

21 \( \left( 1 + 10^{-1000} \right)^{2 \cdot 10^{1000}} \)

**Solution**
\[
\left(1 + 10^{-1000}\right)^{2 \cdot 10^{1000}} = \left(1 + \frac{1}{10^{1000}}\right)^2 \\
\approx e^2 \\
\approx 7.389
\]

23 Estimate the slope of the line containing the points \((5, \ln 5)\) and \((5 + 10^{-100}, \ln(5 + 10^{-100}))\).

**Solution** The slope of the line containing the points \((5, \ln 5)\) and \((5 + 10^{-100}, \ln(5 + 10^{-100}))\) is obtained in the usual way by taking the ratio of the difference of the second coordinates to the difference of the first coordinates:

\[
\frac{\ln(5 + 10^{-100}) - \ln 5}{5 + 10^{-100} - 5} = \frac{\ln\left(1 + \frac{1}{5} \cdot 10^{-100}\right)}{10^{-100}} \\
\approx \frac{\frac{1}{5} \cdot 10^{-100}}{10^{-100}} \\
\approx \frac{1}{5}.
\]

Thus the slope of the line in question is approximately \(\frac{1}{5}\).

25 Suppose \( t \) is a small positive number. Estimate the slope of the line containing the points \((4, e^t)\) and \((4 + t, e^{4+t})\).

**Solution** The slope of the line containing \((4, e^t)\) and \((4 + t, e^{4+t})\) is obtained in the usual way by taking the ratio of the difference of the second coordinates to the difference of the first coordinates:

\[
e^{4+t} - e^t \\
\frac{4 + t - 4}{t} = \frac{e^t (e^t - 1)}{t} \\
\approx \frac{e^t (1 + t - 1)}{t} \\
= \frac{e^t}{t} \\
\approx 54.598
\]

Thus the slope of the line in question is approximately 54.598.

27 Suppose \( r \) is a small positive number. Estimate the slope of the line containing the points \((e^2, 6)\) and \((e^{2+r}, 6 + r)\).

**Solution** The slope of the line containing \((e^2, 6)\) and \((e^{2+r}, 6 + r)\) is obtained in the usual way by taking the ratio of the difference of the second coordinates to the difference of the first coordinates:

\[
\frac{6 + r - 6}{e^{2+r} - e^2} = \frac{r}{e^2 (e^r - 1)} \\
\approx \frac{r}{e^2 (1 + r - 1)} \\
= \frac{1}{e^2} \\
\approx 0.135
\]

Thus the slope of the line in question is approximately 0.135.

29 Find a number \( r \) such that

\[
\left(1 + \frac{r}{10^{90}}\right)^{10^{90}} \approx 5.
\]

**Solution** If \( r \) is not a huge number, then

\[
\left(1 + \frac{r}{10^{90}}\right)^{10^{90}} \approx e^r.
\]

Thus we need to find a number \( r \) such that \( e^r \approx 5 \). This implies that \( r \approx \ln 5 \approx 1.60944 \).

31 Find the number \( c \) such that

\[
\text{area}(\frac{1}{x}, 2, c) = 3.
\]

**Solution** We have

\[
3 = \text{area}(\frac{1}{x}, 2, c) \\
= \text{area}(\frac{1}{x}, 1, c) - \text{area}(\frac{1}{x}, 1, 2) \\
= \ln c - \ln 2 \\
= \ln \frac{c}{2}.
\]

Thus \( \frac{c}{2} = e^3 \), which implies that \( c = 2e^3 \approx 40.171 \).
3.7 **Exponential Growth Revisited**

**LEARNING OBJECTIVES**

By the end of this section you should be able to

- explain the connection between continuous compounding and $e$;
- make computations involving continuous compounding;
- make computations involving continuous growth rates;
- estimate doubling time under continuous compounding.

**Continuously Compounded Interest**

Recall that if interest is compounded $n$ times per year at annual interest rate $r$, then after $t$ years an initial amount $P$ grows to

$$P(1 + \frac{r}{n})^{nt};$$

see Section 3.4 to review the derivation of this formula.

More frequent compounding leads to a larger amount, because interest is earned on the interest more frequently. We could imagine compounding interest once per month ($n = 12$), or once per day ($n = 365$), or once per hour ($n = 365 \times 24 = 8760$), or once per minute ($n = 365 \times 24 \times 60 = 525600$), or once per second ($n = 365 \times 24 \times 60 \times 60 = 31536000$), or even more frequently.

To see what happens when interest is compounded very frequently, we need to consider what happens to the formula above when $n$ is very large. Recall from the previous section that if $n$ is much larger than $r$, then $(1 + \frac{r}{n})^n \approx e^r$. Thus

$$P(1 + \frac{r}{n})^{nt} = P\left((1 + \frac{r}{n})^n\right)^t$$

$$\approx P(e^r)^t$$

$$= Pe^{rt}.$$ 

In other words, if interest is compounded many times per year at annual interest rate $r$, then after $t$ years an initial amount $P$ grows to approximately $Pe^{rt}$. We can think of $Pe^{rt}$ as the amount that we would have if interest were compounded continuously. This formula is actually shorter and cleaner than the formula involving compounding $n$ times per year.

Many banks and other financial institutions use continuous compounding rather than compounding a specific number of times per year. Thus they use the formula derived above involving $e$, which we now restate as follows:

**Continuous compounding**

If interest is compounded continuously at annual interest rate $r$, then after $t$ years an initial amount $P$ grows to

$$Pe^{rt}.$$
Continuous compounding always produces a larger amount than compounding any specific number of times per year. However, for moderate initial amounts, moderate interest rates, and moderate time periods, the difference is not large, as shown in the following example.

**EXAMPLE 1**

Suppose $10,000 is placed in a bank account that pays 5% annual interest.

(a) If interest is compounded continuously, how much will be in the bank account after 10 years?

(b) If interest is compounded four times per year, how much will be in the bank account after 10 years?

**SOLUTION**

(a) The continuous compounding formula shows that $10,000 compounded continuously for 10 years at 5% annual interest grows to become

\[ 10,000e^{0.05 \times 10} \approx 16,487. \]

(b) The compound interest formula shows that $10,000 compounded four times per year for 10 years at 5% annual interest grows to become

\[ 10,000 \left(1 + \frac{0.05}{4}\right)^{4 \times 10} \approx 16,436. \]

**Continuous Growth Rates**

The model presented above of continuous compounding of interest can be applied to any situation with continuous growth at a fixed percentage. The units of time do not necessarily need to be years, but as usual the same time units must be used in all aspects of the model. Similarly, the quantity being measured need not be dollars; for example, this model works well for population growth over time intervals that are not too large.

Because continuous growth at a fixed percentage behaves the same as continuous compounding with money, the formulas are the same. Instead of referring to an annual interest rate that is compounded continuously, we use the term **continuous growth rate**. In other words, the continuous growth rate operates like an interest rate that is continuously compounded.

The continuous growth rate gives a good way to measure how fast something is growing. Again, the magic number \( e \) plays a special role. Our result above about continuous compounding can be restated to apply to more general situations, as follows:

**Continuous growth rates**

If a quantity has a continuous growth rate of \( r \) per unit time, then after \( t \) time units an initial amount \( P \) grows to

\[ Pe^{rt}. \]
Suppose a colony of bacteria has a continuous growth rate of 10% per hour.

(a) By what percent will the colony grow after five hours?

(b) How long will it take for the colony to grow to 250% of its initial size?

**Solution**

(a) A continuous growth rate of 10% per hour means that we should set \( r = 0.1 \).

If the colony starts at size \( P \) at time 0, then at time \( t \) (measured in hours) its size will be \( P e^{0.1t} \).

Thus after five hours the size of the colony will be \( P e^{0.5} \), which is an increase by a factor of \( e^{0.5} \) over the initial size \( P \). Because \( e^{0.5} \approx 1.65 \), this means that the colony will grow by about 65% after five hours.

(b) We want to find \( t \) such that

\[
P e^{0.1t} = 2.5P.
\]

Dividing both sides by \( P \), we see that \( 0.1t = \ln 2.5 \). Thus

\[
t = \frac{\ln 2.5}{0.1} \approx 9.16.
\]

Because \( 0.16 \approx \frac{1}{6} \) and because one-sixth of an hour is 10 minutes, we conclude that it will take about 9 hours, 10 minutes for the colony to grow to 250% of its initial size.

**Doubling Your Money**

The following example shows how to compute how long it takes to double your money with continuous compounding.

**Example 3**

How many years does it take for money to double at 5% annual interest compounded continuously?

**Solution**

After \( t \) years an initial amount \( P \) compounded continuously at 5% annual interest grows to \( P e^{0.05t} \). We want this to equal twice the initial amount. Thus we must solve the equation

\[
P e^{0.05t} = 2P,
\]

which is equivalent to the equation \( e^{0.05t} = 2 \), which implies that \( 0.05t = \ln 2 \). Thus

\[
t = \frac{\ln 2}{0.05} \approx 0.693 \times \frac{69.3}{5} \approx 13.9.
\]

Hence the initial amount of money will double in about 13.9 years.
Suppose we want to know how long it takes money to double at 4% annual interest compounded continuously instead of 5%. Repeating the calculation above, but with 0.04 replacing 0.05, we see that money doubles in about $\frac{69.3}{4}$ years at 4% annual interest compounded continuously. More generally, money doubles in about $\frac{69.3}{R}$ years at $R$ percent interest compounded continuously. Here $R$ is expressed as a percent, rather than as a number. In other words, 5% interest corresponds to $R = 5$.

For quick estimates, usually it is best to round up the 69.3 appearing in the expression $\frac{69.3}{R}$ to 70. Using 70 instead of 69 is easier because 70 is evenly divisible by more numbers than 69 (some people even use 72 instead of 70, but using 70 gives more accurate results than using 72). Thus we have the following useful approximation formula:

**Doubling time**

At $R$ percent annual interest compounded continuously, money doubles in approximately $\frac{70}{R}$ years.

For example, this formula shows that at 5% annual interest compounded continuously, money doubles in about $\frac{70}{5}$ years, which equals 14 years. This estimate of 14 years is close to the more precise estimate of 13.9 years that we obtained above. Furthermore, the computation using the $\frac{70}{R}$ estimate is easy enough to do without a calculator.

Instead of focusing on how long it takes money to double at a specified interest rate, we could ask what interest rate is required to make money double in a specified time period. Here is an example:

What annual interest rate is needed so that money will double in seven years when compounded continuously?

**Solution**

After seven years an initial amount $P$ compounded continuously at $R\%$ annual interest grows to $Pe^{7R/100}$. We want this to equal twice the initial amount. Thus we must solve the equation

$$Pe^{7R/100} = 2P,$$

which is equivalent to the equation $e^{7R/100} = 2$, which implies that

$$\frac{7R}{100} = \ln 2.$$

Thus

$$R = \frac{100 \ln 2}{7} \approx \frac{69.3}{7} \approx 9.9.$$

Hence about 9.9% annual interest will make money double in seven years.
Suppose we want to know what annual interest rate is needed to double money in 11 years when compounded continuously. Repeating the calculation above, but with 11 replacing 7, we see that about \(\frac{69.3}{11}\%\) annual interest would be needed. More generally, we see that to double money in \(t\) years, \(\frac{69.3}{t}\%\) percent interest is needed.

For quick estimates, usually it is best to round up the 69.3 appearing in the expression \(\frac{69.3}{t}\) to 70. Thus we have the following useful approximation formula:

**Doubling rate**

The annual interest rate needed for money to double in \(t\) years with continuous compounding is approximately

\[
\frac{70}{t}
\]

percent.

For example, this formula shows that for money to double in seven years when compounded continuously requires about \(\frac{70}{7}\%\) annual interest, which equals 10%. This estimate of 10% is close to the more precise estimate of 9.9% that we obtained above.

**EXERCISES**

1. How much would an initial amount of $2000, compounded continuously at 6% annual interest, become after 25 years?
2. How much would an initial amount of $3000, compounded continuously at 7% annual interest, become after 15 years?
3. How much would you need to deposit in a bank account paying 4% annual interest compounded continuously so that at the end of 10 years you would have $10,000?
4. How much would you need to deposit in a bank account paying 5% annual interest compounded continuously so that at the end of 15 years you would have $20,000?
5. Suppose a bank account that compounds interest continuously grows from $100 to $110 in two years. What annual interest rate is the bank paying?
6. Suppose a bank account that compounds interest continuously grows from $200 to $224 in three years. What annual interest rate is the bank paying?
7. Suppose a colony of bacteria has a continuous growth rate of 15% per hour. By what percent will the colony have grown after eight hours?
8. Suppose a colony of bacteria has a continuous growth rate of 20% per hour. By what percent will the colony have grown after seven hours?
9. Suppose a country’s population increases by a total of 3% over a two-year period. What is the continuous growth rate for this country?
10. Suppose a country’s population increases by a total of 6% over a three-year period. What is the continuous growth rate for this country?
11. Suppose the amount of the world’s computer hard disk storage increases by a total of 200% over a four-year period. What is the continuous growth rate for the amount of the world’s hard disk storage?
12. Suppose the number of cell phones in the world increases by a total of 150% over a five-year period. What is the continuous growth rate for the number of cell phones in the world?
13. Suppose a colony of bacteria has a continuous growth rate of 30% per hour. If the colony contains 8000 cells now, how many did it contain five hours ago?
14. Suppose a colony of bacteria has a continuous growth rate of 40% per hour. If the colony contains 7500 cells now, how many did it contain three hours ago?
15 Suppose a colony of bacteria has a continuous growth rate of 35% per hour. How long does it take the colony to triple in size?

16 Suppose a colony of bacteria has a continuous growth rate of 70% per hour. How long does it take the colony to quadruple in size?

17 About how many years does it take for money to double when compounded continuously at 2% per year?

18 About how many years does it take for money to double when compounded continuously at 10% per year?

19 About how many years does it take for $200 to become $800 when compounded continuously at 2% per year?

20 About how many years does it take for $300 to become $2,400 when compounded continuously at 5% per year?

21 How long does it take for money to triple when compounded continuously at 5% per year?

22 How long does it take for money to increase by a factor of five when compounded continuously at 7% per year?

23 Find a formula for estimating how long money takes to triple at $R$ percent annual interest rate compounded continuously.

24 Find a formula for estimating how long money takes to increase by a factor of ten at $R$ percent annual interest compounded continuously.

25 Suppose one bank account pays 5% annual interest compounded once per year, and a second bank account pays 5% annual interest compounded continuously. If both bank accounts start with the same initial amount, how long will it take for the second bank account to contain twice the amount of the first bank account?

26 Suppose one bank account pays 3% annual interest compounded once per year, and a second bank account pays 4% annual interest compounded continuously. If both bank accounts start with the same initial amount, how long will it take for the second bank account to contain 50% more than the first bank account?

27 Suppose a colony of 100 bacteria cells has a continuous growth rate of 30% per hour. Suppose a second colony of 200 bacteria cells has a continuous growth rate of 20% per hour. How long does it take for the two colonies to have the same number of bacteria cells?

28 Suppose a colony of 50 bacteria cells has a continuous growth rate of 35% per hour. Suppose a second colony of 300 bacteria cells has a continuous growth rate of 15% per hour. How long does it take for the two colonies to have the same number of bacteria cells?

29 Suppose a colony of bacteria has doubled in five hours. What is the approximate continuous growth rate of this colony of bacteria?

30 Suppose a colony of bacteria has doubled in two hours. What is the approximate continuous growth rate of this colony of bacteria?

31 Suppose a colony of bacteria has tripled in five hours. What is the continuous growth rate of this colony of bacteria?

32 Suppose a colony of bacteria has tripled in two hours. What is the continuous growth rate of this colony of bacteria?

33 Using compound interest, explain why

\[
(1 + \frac{0.05}{n})^n < e^{0.05}
\]

for every positive integer $n$.

34 Suppose in Exercise 9 we had simply divided the 3% increase over two years by 2, getting 1.5% per year. Explain why this number is close to the more accurate answer of approximately 1.48% per year.

35 Suppose in Exercise 11 we had simply divided the 200% increase over four years by 4, getting 50% per year. Explain why we should not be surprised that this number is not close to the more accurate answer of approximately 27.5% per year.

36 Explain why every function $f$ with exponential growth (see Section 3.4 for the definition) can be written in the form

\[
f(x) = ce^{kx},
\]

where $c$ and $k$ are positive constants.
37 In Section 3.4 we saw that if a population doubles every $d$ time units, then the function $p$ modeling this population growth is given by the formula

\[ p(t) = p_0 \cdot 2^{t/d}, \]

where $p_0$ is the population at time 0. Some books do not use the formula above but instead use the formula

\[ p(t) = p_0 e^{(t \ln 2)/d}. \]

Show that the two formulas above are really the same. [Which of the two formulas in this problem do you think is cleaner and easier to understand?]

WORKED-OUT SOLUTIONS to Odd-Numbered Exercises

1 How much would an initial amount of $2000, compounded continuously at 6% annual interest, become after 25 years?

**SOLUTION** After 25 years, $2000 compounded continuously at 6% annual interest would grow to $2000e^{0.06\times25}$ dollars, which equals $2000e^{1.5}$ dollars, which is approximately $8963.

3 How much would you need to deposit in a bank account paying 4% annual interest compounded continuously so that at the end of 10 years you would have $10,000?

**SOLUTION** We need to find $P$ such that

\[ 10000 = Pe^{0.04\times10} = Pe^{0.4}. \]

Thus

\[ P = \frac{10000}{e^{0.4}} \approx 6703. \]

In other words, the initial amount in the bank account should be $\frac{10000}{e^{0.4}}$ dollars, which is approximately $6703.

5 Suppose a bank account that compounds interest continuously grows from $100 to $110 in two years. What annual interest rate is the bank paying?

**SOLUTION** Let $r$ denote the annual interest rate paid by the bank. Then

\[ 110 = 100e^{2r}. \]

Dividing both sides of this equation by 100 gives $1.1 = e^{2r}$, which implies that $2r = \ln 1.1$, which is equivalent to

\[ r = \frac{\ln 1.1}{2} \approx 0.0477. \]

Thus the annual interest is approximately 4.77%.

7 Suppose a colony of bacteria has a continuous growth rate of 15% per hour. By what percent will the colony have grown after eight hours?

**SOLUTION** A continuous growth rate of 15% per hour means that $r = 0.15$. If the colony starts at size $P$ at time 0, then at time $t$ (measured in hours) its size will be $Pe^{0.15t}$.

Because $0.15 \times 8 = 1.2$, after eight hours the size of the colony will be $Pe^{1.2}$, which is an increase by a factor of $e^{1.2}$ over the initial size $P$. Because $e^{1.2} \approx 3.32$, this means that the colony will be about 332% of its original size after eight hours. Thus the colony will have grown by about 232% after eight hours.

9 Suppose a country’s population increases by a total of 3% over a two-year period. What is the continuous growth rate for this country?

**SOLUTION** A 3% increase means that we have $1.03$ times as much as the initial amount. Thus $1.03P = Pe^{2r}$, where $P$ is the country’s population at the beginning of the measurement period and $r$ is the country’s continuous growth rate. Thus $e^{2r} = 1.03$, which means that $2r = \ln 1.03$. Thus $r = \frac{\ln 1.03}{2} \approx 0.0148$. Thus the country’s continuous growth rate is approximately 1.48% per year.

11 Suppose the amount of the world’s computer hard disk storage increases by a total of 200% over a four-year period. What is the continuous growth rate for the amount of the world’s hard disk storage?

**SOLUTION** A 200% increase means that we have three times as much as the initial amount. Thus $3P = Pe^{4r}$, where $P$ is amount of the world’s hard disk storage at the beginning of the measurement period and $r$
Suppose a colony of bacteria has a continuous growth rate of 30% per hour. If the colony contains 8000 cells now, how many did it contain five hours ago?

**SOLUTION** Let $P$ denote the number of cells at the initial time five hours ago. Thus we have $8000 = Pe^{0.3 \times 5}$, or $8000 = Pe^{1.5}$. Thus

$$P = \frac{8000}{e^{1.5}} \approx 1785.$$

Suppose a colony of bacteria has a continuous growth rate of 35% per hour. How long does it take for the colony to triple in size?

**SOLUTION** Let $P$ denote the initial size of the colony, and let $t$ denote the time that it takes the colony to triple in size. Then $3P = Pe^{0.35t}$, which means that $e^{0.35t} = 3$. Thus $0.35t = \ln 3$, which implies that $t = \frac{\ln 3}{0.35} \approx 3.14$. Thus the colony triples in size in approximately 3.14 hours.

About how many years does it take for money to double when compounded continuously at 2% per year?

**SOLUTION** At 2% per year compounded continuously, money will double in approximately $\frac{70}{2}$ years, which equals 35 years.

About how many years does it take for $200 to become $800 when compounded continuously at 2% per year?

**SOLUTION** At 2% per year, money doubles in approximately 35 years. For $200 to become $800, it must double twice. Thus this will take about 70 years.

How long does it take for money to triple when compounded continuously at 5% per year?

**SOLUTION** To triple an initial amount $P$ in $t$ years at 5% annual interest compounded continuously, the following equation must hold:

$$Pe^{0.05t} = 3P.$$

Dividing both sides by $P$ and then taking the natural logarithm of both sides gives $0.05t = \ln 3$. Thus $t = \frac{\ln 3}{0.05}$. Thus it would take $\frac{\ln 3}{0.05}$ years, which is about 22 years.

Find a formula for estimating how long money takes to triple at $R$ percent annual interest rate compounded continuously.

**SOLUTION** To triple an initial amount $P$ in $t$ years at $R$ percent annual interest compounded continuously, the following equation must hold:

$$Pe^{Rt/100} = 3P.
$$

Dividing both sides by $P$ and then taking the natural logarithm of both sides gives $Rt/100 = \ln 3$. Thus $t = \frac{100\ln 3}{R}$. Because $\ln 3 \approx 1.10$, this shows that money triples in about $\frac{110}{R}$ years.

Suppose one bank account pays 5% annual interest compounded once per year, and a second bank account pays 5% annual interest compounded continuously. If both bank accounts start with the same initial amount, how long will it take for the second bank account to contain twice the amount of the first bank account?

**SOLUTION** Suppose both bank accounts start with $P$ dollars. After $t$ years, the first bank account will contain $P(1.05)^t$ dollars and the second bank account will contain $Pe^{0.05t}$ dollars. Thus we need to solve the equation

$$\frac{Pe^{0.05t}}{P(1.05)^t} = 2.$$

The initial amount $P$ drops out of this equation (as expected), and we can rewrite this equation as follows:

$$2 = \frac{e^{0.05t}}{1.05^t} = \left(\frac{e^{0.05}}{1.05}\right)^t.$$

Taking the natural logarithm of the first and last terms above gives

$$\ln 2 = t \ln \frac{e^{0.05}}{1.05} = t (\ln e^{0.05} - \ln 1.05) = t (0.05 - \ln 1.05),$$

which we can then solve for $t$, getting

$$t = \frac{\ln 2}{0.05 - \ln 1.05}.$$

Using a calculator to evaluate the expression above, we see that $t$ is approximately 573 years.

Suppose a colony of 100 bacteria cells has a continuous growth rate of 30% per hour. Suppose a second colony of 200 bacteria cells has a continuous growth rate of 20% per hour. How long does it take for the two colonies to have the same number of bacteria cells?
SOLUTION After $t$ hours, the first colony contains $100e^{0.3t}$ bacteria cells and the second colony contains $200e^{0.2t}$ bacteria cells. Thus we need to solve the equation

$$100e^{0.3t} = 200e^{0.2t}.$$ 

Dividing both sides by 100 and then dividing both sides by $e^{0.2t}$ gives the equation

$$e^{0.1t} = 2.$$ 

Thus $0.1t = \ln 2$, which implies that

$$t = \frac{\ln 2}{0.1} \approx 6.93.$$ 

Thus the two colonies have the same number of bacteria cells in a bit less than 7 hours.

29 Suppose a colony of bacteria has doubled in five hours. What is the approximate continuous growth rate of this colony of bacteria?

SOLUTION The approximate formula for doubling the number of bacteria is the same as for doubling money. Thus if a colony of bacteria doubles in five hours, then it has a continuous growth rate of approximately $(70/5)\%$ per hour. In other words, this colony of bacteria has a continuous growth rate of approximately 14\% per hour.

31 Suppose a colony of bacteria has tripled in five hours. What is the continuous growth rate of this colony of bacteria?

SOLUTION Let $r$ denote the continuous growth rate of this colony of bacteria. If the colony initially contains $P$ bacteria cells, then after five hours it will contain $Pe^{5r}$ bacteria cells. Thus we need to solve the equation

$$Pe^{5r} = 3P.$$ 

Dividing both sides by $P$ gives the equation $e^{5r} = 3$, which implies that $5r = \ln 3$. Thus

$$r = \frac{\ln 3}{5} \approx 0.2197.$$ 

Thus the continuous growth rate of this colony of bacteria is approximately 22\% per hour.
CHAPTER SUMMARY

To check that you have mastered the most important concepts and skills covered in this chapter, make sure that you can do each item in the following list:

- Define logarithms.
- Use common logarithms to determine how many digits a number has.
- Use the formula for the logarithm of a power.
- Model radioactive decay using half-life.
- Use the change-of-base formula for logarithms.
- Use the formulas for the logarithm of a product and a quotient.
- Use logarithmic scales for measuring earthquakes, sound, and stars.
- Model population growth using functions with exponential growth.
- Compute compound interest.
- Approximate the area under a curve using rectangles.
- Explain the definition of $e$.
- Explain the definition of the natural logarithm.
- Approximate $e^x$ and $\ln(1 + x)$ for $|x|$ small.
- Compute continuously compounded interest.
- Estimate how long it takes to double money at a given interest rate.

To review a chapter, go through the list above to find items that you do not know how to do, then reread the material in the chapter about those items. Then try to answer the chapter review questions below without looking back at the chapter.

CHAPTER REVIEW QUESTIONS

1. Find a formula for $(f \circ g)(x)$, where $f(x) = 3x^{1/2}$ and $g(x) = x^{1/3}$.
2. Explain how logarithms are defined.
3. Explain why logarithms with base 0 are not defined.
4. What is the domain of the function $f$ defined by $f(x) = \log_2(5x + 1)$?
5. What is the range of the function $f$ defined by $f(x) = \log_7 x$?
6. Explain why $3^{\log_3 7} = 7$.
7. Explain why $\log_5(5^{444}) = 444$.
8. Without using a calculator or computer, estimate the number of digits in $2^{1000}$.
9. Find a number $x$ such that $\log_3(4^x + 1) = 2$.
10. Find all numbers $x$ such that $\log x + \log(x + 2) = 1$.
11. Evaluate $\log_5 \sqrt{25}$.
12. Find a number $b$ such that $\log_b 9 = -2$.
13. How many digits does $4^{7000}$ have?
14. At the time this book was written, the largest known prime number not of the form $2^n - 1$ was $19249 \cdot 2^{13018586} + 1$. How many digits does this prime number have?
15. Find the smallest integer $m$ such that $8^m > 10^{500}$.
16. Find the largest integer $k$ such that $15^k < 11^{900}$.
17. Explain why $\log 200 = 2 + \log 2$.
18. Explain why $\log \sqrt{300} = 1 + \frac{\log 3}{2}$.
19. Which of the expressions
   $$\log x - \log y$$
   and
   $$\frac{\log x}{\log y}$$
   can be rewritten using only one log?
20. Find a formula for the inverse of the function $f$ defined by $f(x) = 4 + 5\log_3(7x + 2)$.
21. Find a formula for $(f \circ g)(x)$, where $f(x) = 7^{4x}$ and $g(x) = \log_7 x$.
22. Find a formula for $(f \circ g)(x)$, where $f(x) = \log_2 x$ and $g(x) = 2^{5x - 9}$.
23. Evaluate $\log_{3.2} 456$.
24. Suppose $\log_6 t = 4.3$. Evaluate $\log_6(t^{200})$. 
25 Suppose \( \log_7 w = 3.1 \) and \( \log_7 z = 2.2 \). Evaluate \( \log_7 \left( \frac{49w^2}{z^3} \right) \).

26 Suppose \$7000 is deposited in a bank account paying 4% interest per year, compounded 12 times per year. How much will be in the bank account at the end of 50 years?

27 Suppose \$5000 is deposited in a bank account that compounds interest four times per year. The bank account contains \$9900 after 13 years. What is the annual interest rate for this bank account?

28 A colony that initially contains 100 bacteria cells is growing exponentially, doubling in size every 75 minutes. Approximately how many bacteria cells will the colony have after 6 hours?

29 A colony of bacteria is growing exponentially, doubling in size every 50 minutes. How many minutes will it take for the colony to become six times its current size?

30 A colony of bacteria is growing exponentially, increasing in size from 200 to 500 cells in 100 minutes. How many minutes does it take the colony to double in size?

31 Explain why a population cannot have exponential growth indefinitely.

32 How many years will it take for a sample of cesium-137 (half-life of 30 years) to have only 3% as much cesium-137 as the original sample?

33 How many more times intense is an earthquake with Richter magnitude 6.8 than an earthquake with Richter magnitude 6.1?

34 Explain why adding ten decibels to a sound multiplies the intensity of the sound by a factor of 10.

35 Most stars have an apparent magnitude that is a positive number. However, four stars (not counting the sun) have an apparent magnitude that is a negative number. Explain how a star can have a negative apparent magnitude.

36 What is the definition of \( e \)?

37 What are the domain and range of the function \( f \) defined by \( f(x) = e^x \)?

38 What is the definition of the natural logarithm?

39 What are the domain and range of the function \( g \) defined by \( g(y) = \ln y \)?

40 Find a number \( t \) such that \( \ln(4t + 3) = 5 \).

41 What number \( t \) makes \( (e^{t+8})^t \) as small as possible?

42 Find a number \( w \) such that \( e^{2w-7} = 6 \).

43 Find a formula for the inverse of the function \( g \) defined by \( g(x) = 8 - 3e^{3x} \).

44 Find a formula for the inverse of the function \( h \) defined by \( h(x) = 1 - \ln(x + 4) \).

45 Find the area of the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = 1 \) and \( x = e^2 \).

46 Find a number \( c \) such that the area of the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = 3 \) and \( x = 5 \) is 45.

47 What is the area of the region under the curve \( y = \frac{1}{x} \), above the \( x \)-axis, and between the lines \( x = 1 \) and \( x = c \) is 45.

48 Draw an appropriate figure and use it to explain why \( \ln(1.0001) \approx 0.0001 \).

49 Estimate the slope of the line containing the points \((2, \ln(6 + 10^{-500}))\) and \((6, \ln 6)\).

50 Estimate the value of \( \frac{e^{1000.002}}{e^{1000}} \).

51 Estimate the slope of the line containing the points \((6, e^{0.0002})\) and \((2, 1)\).

52 Estimate the value of \( \left( 1 - \frac{6}{788} \right)^{288} \).

53 How much would an initial amount of \$12,000, compounded continuously at 6% annual interest, become after 20 years?

54 How much would you need to deposit in a bank account paying 6% annual interest compounded continuously so that at the end of 25 years you would have \$100,000?

55 A bank account that compounds interest continuously grows from \$2000 to \$2878.15 in seven years. What annual interest rate is the bank paying?

56 Approximately how many years does it take for money to double when compounded continuously at 5% per year?

57 Suppose a colony of bacteria has doubled in 10 hours. What is the approximate continuous growth rate of this colony of bacteria?

58 At 5% interest compounded continuously but paid monthly (as many banks do), how long will it take to turn a hundred dollars into a million dollars? (See the Dilbert comic in Section 3.4.)