Preface to the Instructor

Goals and Prerequisites

This book seeks to prepare students to succeed in calculus. Thus this book focuses on topics that students need for calculus, especially first-semester calculus. Many important subjects that should be known by all educated citizens but that are irrelevant to calculus have been excluded.

Precalculus is a one-semester course at most universities. Nevertheless, typical precalculus textbooks contain about a thousand pages (not counting a student solutions manual), far more than can be covered in one semester. By emphasizing topics crucial to success in calculus, this book has a more manageable size even though it includes a student solutions manual. A thinner textbook should indicate to students that they are truly expected to master most of the content of the book.

The prerequisite for this course is the usual course in intermediate algebra. Many students in precalculus classes have had a trigonometry course previously, but this book does not assume that the students remember any trigonometry. In fact the book is fairly self-contained, starting with a review of the real numbers in Chapter 0, whose numbering is intended to indicate that many instructors will prefer to skip this beginning material or cover it quickly.

Chapter 0 could have been titled **A Prelude to A Prelude to Calculus**.

Inverse Functions

The unifying concept of inverse functions are introduced early in the book in Section 1.5. This crucial idea has its first major use in this book in the definition of $y^{1/m}$ as the number that when raised to the m^{th} power gives y (in other words, the function $y \mapsto y^{1/m}$ is the inverse of the function $x \mapsto x^m$; see Section 3.1). The second major use of inverse functions occurs in the definition of $\log_b y$ as the number such that b raised to this number gives y (in other words, the function $y \mapsto \log_b y$ is the inverse of the function $x \mapsto b^x$; see Section 3.2).

Thus students should be comfortable with using inverse functions by the time they reach the inverse trigonometric functions (arccosine, arcsine, and arctangent) in Section 5.7. This familiarity with inverse functions should help students deal with inverse operations (such as antidifferentiation) when they reach calculus.

Algebraic Properties of Logarithms

Logarithms play a key role in calculus, but many calculus instructors complain that too many students lack appropriate algebraic manipulation skills with logarithms. In Chapter 3 logarithms are defined as the inverse functions of exponentiation. The base for logarithms here is arbitrary, although most of the examples and motivation in Chapter 3 use logarithms base 2 or logarithms base 10. In Chapter 3, the algebraic properties of logarithms are easily derived from the algebraic properties of exponentiation.

The initial separation of logarithms and e should help students master both concepts.

The crucial concepts of e and natural logarithms are saved for a later chapter. Thus students can concentrate in Chapter 3 on understanding logarithms (arbitrary base) and their properties without at the same time worrying about grasping concepts related to e. Similarly, when natural logarithms arise naturally in Chapter 4, students should be able to concentrate on issues surrounding e without at the same time learning properties of logarithms.

Half-life and Exponential Growth

All precalculus textbooks present radioactive decay as an example of exponential decay. Amazingly, the typical precalculus textbook states that if a radioactive isotope has a half-life of h, then the amount left at time t will equal e^{-kt} times the amount present at time 0, where $k = \frac{\ln 2}{h}$.

A much clearer formulation would state, as this textbook does, that the amount left at time t will equal $2^{-t/h}$ times the amount present at time 0. The unnecessary use of e and $\ln 2$ in this context may suggest to students that e and natural logarithms have only contrived and artificial uses, which is not the message that students should receive from their textbook. Using $2^{-t/h}$ helps students understand the concept of half-life, with a formula connected to the meaning of the concept.

Similarly, many precalculus textbooks consider, for example, a colony of bacteria doubling in size every 3 hours, with the textbook then producing the formula $e^{(t \ln 2)/3}$ for the growth factor after t hours. The simpler and natural formula $2^{t/3}$ seems not to be mentioned in such books. This book presents the more natural approach to such issues of exponential growth and decay.

Area

About half of calculus (namely, integration) deals with area, but most precalculus textbooks barely mention the subject.

Chapter 4 in this book builds the intuitive notion of area starting with squares, and then quickly derives formulas for the area of rectangles, triangles, parallelograms, and trapezoids. A discussion of the effects of stretching either horizontally or vertically easily leads to the familiar formula for the area enclosed by a circle. Chapter 4 uses the same ideas to derive the formula for the area inside an ellipse.

Chapter 4 then turns to the question of estimating the area under parts of the curve $y = \frac{1}{x}$ by using rectangles. This easy nontechnical introduction, with its emphasis on ideas without the clutter of the notation of Riemann sums, will serve students well when they reach integral calculus in a later course.

e, The Exponential Function, and the Natural Logarithm

Most precalculus textbooks either present no motivation for e or motivate e via continuously compounding interest or through the limit of an indeterminate expression of the form 1^∞; these concepts are difficult for students at this level to understand.

Chapter 4 presents a clean and well-motivated approach to e and the natural logarithm. We do this by looking at the area (intuitively defined) under the curve $y = \frac{1}{x}$, above the x-axis, and between the lines $x = 1$ and $x = c$.

A similar approach to e and the natural logarithm is common in calculus courses. However, this approach is not usually adopted in precalculus textbooks. Using obvious properties of area, the simple presentation given here shows how these ideas can come through clearly without the technicalities of calculus or Riemann sums. Indeed, this precalculus approach to the exponential function and the natural logarithm shows that a good understanding of these subjects need not wait until the calculus course. Students who have seen the approach given here should be well prepared to deal with these concepts in their calculus courses.

The approach taken here also has the advantage that it easily leads, as we will see in Chapter 4, to the approximation $\ln(1+h) \approx h$ for small values of h. Furthermore, the same methods show that if r is any number, then $(1+\frac{r}{x})^x \approx e^r$ for large values of x. A final bonus of this approach is that the connection between continuously compounding interest and e becomes a nice corollary of natural considerations concerning area.

Trigonometry

Should the trigonometric functions be introduced via the unit circle or via right triangles? Calculus requires the unit-circle approach (because, for example, discussing the Taylor series for $\cos x$ requires us to consider negative values of x and values of x that are more than $\frac{\pi}{2}$ radians). Thus this textbook uses the unit-circle approach, but quickly gives applications to right triangles. The unit-circle approach also allows for a well-motivated introduction to radian measure.

The trigonometry section of this book concentrates almost exclusively on the functions cosine, sine, and tangent and their inverse functions, with only cursory mention of secant, cosecant, and cotangent. These latter three functions, which are simply the multiplicative inverses of the three key trigonometric functions, add little content or understanding.

Exercises and Problems

Each exercise has a unique correct answer, usually a number or a function; each problem has multiple correct answers, usually explanations or examples.

Students learn mathematics by actively working on a wide range of exercises and problems. Ideally, a student who reads and understands the material in a section of this book should be able to do the exercises and problems in that section without further help. However, some of the exercises require application of the ideas in a context that students may not have seen before, and many students will need help with these exercises. This help is available from the complete worked-out solutions to all the odd-numbered exercises that appear at the end of each section.

Because the worked-out solutions were written solely by the author of the textbook, students can expect a consistent approach to the material. Furthermore, students will save money by not having to purchase a separate student solutions manual.

This book contains what is usually a separate book called the student solutions manual.

The exercises (but not the problems) occur in pairs, so that an odd-numbered exercise is followed by an even-numbered exercise whose solution uses the same ideas and techniques. A student stumped by an even-numbered exercise should be able to tackle it after reading the worked-out solution to the corresponding odd-numbered exercise. This arrangement allows the text to focus more centrally on explanations of the material and examples of the concepts.

My experience with teaching precalculus is that most students read the student solutions manual when they are assigned homework, even though they are reluctant to read the main text. The integration of the student solutions manual within this book might encourage students who would otherwise read only the student solutions manual to drift over and also read the main text. To reinforce this tendency, the worked-out solutions to the odd-numbered exercises at the end of each section are typeset with a slightly less appealing style (smaller type, two-column format, and not right justified) than the main text. The reader-friendly appearance of the main text might nudge students to spend some time there.

Exercises and problems in this book vary greatly in difficulty and purpose. Some exercises and problems are designed to hone algebraic manipulation skills; other exercises and problems are designed to push students to genuine understanding beyond rote algorithmic calculation.

Some exercises and problems intentionally reinforce material from earlier in the book. For example, Exercise 27 in Section 5.3 asks students to find the smallest number x such that $\sin(e^x) = 0$; students will need to understand that they want to choose x so that $e^x = \pi$ and thus $x = \ln \pi$. Although such exercises require more thought than most exercises in the book, they allow students to see crucial concepts more than once, sometimes in unexpected contexts.

A Book Designed to be Read

Mathematics faculty frequently complain, with justification, that most students in lower-division mathematics courses do not read the textbook. When doing homework, a typical precalculus student looks only at the relevant section of the textbook or the student solutions manual for an example similar to the homework problem at hand. The student reads enough of that example to imitate the procedure, does the homework problem, and then follows the same process with the next homework problem. Little understanding may take place.

In contrast, this book is designed to be read by students. The writing style and layout are meant to induce students to read and understand the material. Explanations are more plentiful than typically found in precalculus books, with examples of the concepts making the ideas concrete whenever possible.

The Calculator Issue

To aid instructors in presenting the kind of course they want, the symbol  *appears with exercises and problems that require students to use a calculator.*

The issue of whether and how calculators should be used by students has generated immense controversy.

Some sections of this book have many exercises and problems designed for calculators (for example Section 3.4 on exponential growth and Section 6.2 on the law of sines and the law of cosines), but some sections deal with material not as amenable to calculator use. The emphasis throughout the text has been on giving students both the understanding and the skills they need to succeed in calculus. Thus the book does not aim for an artificially predetermined percentage of exercises and problems in each section requiring calculator use.

Some exercises and problems that require a calculator are intentionally designed to make students realize that by understanding the material, they can overcome the limitations of calculators. As one example among many, Exercise 41 in Section 3.3 asks students to find the number of digits in the decimal expansion of 7^{4000}. Brute force with a calculator will not work with this problem because the number involved has too many digits. However, a few moments' thought should show students that they can solve this problem by using logarithms (and their calculators!).

The calculator icon can be interpreted for some exercises, depending on the instructor's preference, to mean that the solution should be a decimal approximation rather than the exact answer. For example, Exercise 3 in Section 4.5 asks how much would need to be deposited in a bank account paying 4% interest compounded continuously so that at the end of 10 years the account would contain \$10,000. The exact answer to this exercise is $10000/e^{0.4}$ dollars, but it may be more satisfying to the student (after obtaining the exact answer) to use a calculator to see that approximately \$6,703 needs to be deposited. For such exercises, instructors can decide whether to ask for exact answers or decimal approximations (the worked-out solutions for the odd-numbered exercises will usually contain both).

Regardless of what level of calculator use an instructor expects, students should not turn to a calculator to compute something like $\cos 0$*, because then* $\cos$ *has become just a button on the calculator.*

What to Cover

Different instructors will want to cover different sections of this book. I usually cover Chapter 0 (The Real Numbers), even though it should be review, because it deals with familiar topics in a deeper fashion than students may have previously seen. I frequently cover Section 2.5 (Rational Functions) only lightly because graphing rational functions, and in particular finding local minima and maxima, is better done with calculus. Many instructors will prefer to skip Chapter 7 (Sequences, Series, and Limits), leaving that material to a calculus course. A one-semester precalculus course will probably not have time to cover the sections denoted with an asterisk (*); those sections can safely be skipped by courses focusing only on material needed for first-semester calculus.

What's New as Compared to the Preliminary Edition

Numerous improvements have been made throughout the text based upon suggestions from faculty and students who used the Preliminary Edition. For example, the introduction to e now gives instructors a gentler path to help lead students to discover this remarkable number. More exercises and problems have been added to many sections.

Some faculty requested coverage of additional topics because their precalculus courses serve other purposes beyond preparing students for first-semester calculus. Thus three new optional sections have been added, dealing with complex numbers, systems of equations and matrices, and vectors.

A major redesign using full color has led to considerable improvements in the appearance of the book.

Finally, a comprehensive index now allows users to locate topics within the book quickly.

Comments Welcome

I seek your help in making this a better book. Please send me your comments and your suggestions for improvements. Thanks!

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