Preface to the Instructor

Goals and Prerequisites

This book seeks to prepare students to succeed in calculus. Thus it focuses on topics that students need for calculus, especially first-semester calculus. Important parts of mathematics that should be known by all educated citizens but that are irrelevant to calculus have been excluded.

Precalculus is a one-semester course at most colleges and universities. Nevertheless, typical precalculus textbooks contain about a thousand pages (not counting a student solutions manual), far more than can be covered in one semester.

By emphasizing topics crucial to success in calculus, this book has a more manageable size even though it includes a student solutions manual. A thinner textbook should indicate to students that they are truly expected to master most of the content of the book.

The prerequisite for this course is the usual course in intermediate algebra. Many students in precalculus classes have had a trigonometry course previously, but this book does not assume students remember any trigonometry. The book is fairly self-contained, starting with a review of the real numbers in Chapter 0, whose numbering is intended to indicate that many instructors will prefer to cover this beginning material quickly or skip it.

Chapter 0 could have been titled **A Prelude to A Prelude to Calculus**.

Different instructors will want to cover different sections of this book. My personal preference is to finish up through Section 6.2 (*Series*). By including the sections on sequences and series, you will give students some experience with using subscript notation and summation notation that will be useful when they get to Riemann integration. The last chapter (*Polar Coordinates, Vectors, and Complex Numbers*) deals with topics that are typically more useful for second-semester calculus. I do not cover this chapter when teaching precalculus because I prefer to focus on getting students ready to succeed in first-semester calculus. Other instructors have different preferences, which is why I have included the last chapter in this book.

A Book Designed to be Read

Mathematics faculty frequently complain, with justification, that most students in lower-division mathematics courses do not read the textbook.

When doing homework, a typical precalculus student looks only at the relevant section of the textbook or the student solutions manual for an example similar to the homework exercise at hand. The student reads enough of that example to imitate the procedure, does the homework exercise, and then follows the same process with the next homework exercise. Little understanding may take place.

In contrast, this book is designed to be read by students. The writing style and layout are meant to induce students to read and understand the material. Explanations are more plentiful than typically found in precalculus books, with examples of the concepts making the ideas concrete whenever possible.

The text often points out to students that understanding the material will be more useful than memorizing it.

As a visual aid to students, boxes in this book are color-coded to show their function. Specifically, boxes with yellow shading give definitions, and boxes with blue shading give results (which in many books are called theorems or corollaries).

Exercises and Problems

Each exercise in this book has a unique correct answer, usually a number or a function. Each problem in this book has multiple correct answers, usually consisting of explanations or examples.

Students learn mathematics by actively working on a wide range of exercises and problems. Ideally, a student who reads and understands the material in a section of this book should be able to do the exercises and problems in that section without further help. However, some of the exercises require application of the ideas in a context that students may not have seen before, and many students will need help with these exercises. This help is available from the complete worked-out solutions to all the odd-numbered exercises that appear at the end of each section.

Because the worked-out solutions were written solely by the author of the textbook, students can expect an unusually consistent approach to the material. Students will be happy to save money by not having to purchase a separate student solutions manual.

This book contains what is usually a separate book called the student solutions manual. Thus it is even thinner in comparison to competing bloated books than is indicated by just a page count.

The exercises (but not the problems) occur in pairs, so that an odd-numbered exercise is followed by an even-numbered exercise whose solution uses the same ideas and techniques. A student stumped by an even-numbered exercise should be able to tackle it after reading the worked-out solution to the corresponding odd-numbered exercise. This arrangement allows the text to focus more centrally on explanations of the material and examples of the concepts.

Many students read the student solutions manual when they are assigned homework, even though they may be reluctant to read the main text. The integration of the student solutions manual within this book may encourage students who would otherwise read only the student solutions manual to drift over and also read the main text. To reinforce this tendency, the worked-out solutions to the odd-numbered exercises at the end of each section are typeset in a slightly less appealing style (smaller type and two-column format) than the main text. The reader-friendly appearance of the main text may nudge students to spend some time there.

Exercises and problems in this book vary greatly in difficulty and purpose. Some exercises and problems are designed to hone algebraic manipulation skills; other exercises and problems are designed to push students to genuine understanding beyond rote algorithmic calculation.

Some exercises and problems intentionally reinforce material from earlier in the book. For example, Exercise 27 in Section 4.3 asks students to find the smallest number x such that $\sin(e^x) = 0$; students will need to understand that they want to choose x so that $e^x = \pi$ and thus $x = \ln \pi$. Although such exercises require more thought than most exercises in the book, they allow students to see crucial concepts more than once, sometimes in unexpected contexts.

For instructors who want to offer online grading to their students, exercises from this book are available via either *WileyPLUS* or *WebAssign*. These online learning systems give students instant feedback and keep records for instructors. Most of the exercises in this book have been translated into algorithmically generated exercises in these two online learning systems, creating an essentially unlimited number of variations. These systems give instructors the flexibility of allowing students who answer an exercise incorrectly to attempt similar exercises requiring the same ideas and techniques.

The Calculator Issue

To aid instructors in presenting the kind of course they want, the symbol  *appears with exercises and problems that require students to use a calculator.*

The issue of whether and how calculators should be used by students has generated immense controversy.

Some sections of this book have many exercises and problems designed for calculators—examples include Section 3.4 on exponential growth and Section 5.4 on the law of sines and the law of cosines. However, some sections deal with material not as amenable to calculator use. Throughout the text, the emphasis is on giving students both the understanding and the skills they need to succeed in calculus. Thus the book does not aim for an artificially predetermined percentage of exercises and problems in each section requiring calculator use.

Some exercises and problems that require a calculator are intentionally designed to make students realize that by understanding the material, they can overcome the limitations of calculators. For example, Exercise 15 in Section 3.2 asks students to find the number of digits in the decimal expansion of 7^{4000}. Brute force with a calculator will not work with this problem because the number involved has too many digits. However, a few moments' thought should show students that they can solve this problem by using logarithms (and their calculators!).

The calculator icon can be interpreted for some exercises, depending on the instructor's preference, to mean that the solution should be a decimal approximation rather than the exact answer. For example, Exercise 3 in Section 3.7 asks how much would need to be deposited in a bank account paying 4% interest compounded continuously so that at the end of 10 years the account would contain \$10,000. The exact answer to this exercise is $10000/e^{0.4}$ dollars, but it may be more satisfying to the student (after obtaining the exact answer) to use a calculator to see that approximately \$6,703 needs to be deposited. For such exercises, instructors can decide whether to ask for exact answers or decimal approximations (the worked-out solutions for the odd-numbered exercises usually contain both types of solutions).

Regardless of what level of calculator use an instructor expects, students should not turn to a calculator to compute something like $\cos 0$*, because then* cos *has become just a button on the calculator.*

Functions

In preparation for writing this book, I asked many calculus instructors what improvements they would like to see in the preparation of their calculus students. The two most common answers I received were (1) better understanding of functions and (2) better algebraic manipulation skills. Both of these goals are intertwined throughout the book.

Because of the importance of functions, Chapter 1 (*Functions and Their Graphs*) is devoted to functions, considerably earlier than in many precalculus books. Particular attention is paid to function transformations, composition of functions, and inverse functions.

A good understanding of the composition of functions will be tremendously useful to students when they get to the chain rule in calculus.

The unifying concept of inverse functions appears several times later in the book. In particular, $y^{1/m}$ is defined as the number that when raised to the m^{th} power gives y (in other words, the function $y \mapsto y^{1/m}$ is the inverse of the function $x \mapsto x^m$; see Section 2.3). Later, a second major use of inverse functions occurs in the definition of $\log_b y$ as the number such that b raised to this number gives y (in other words, the function $y \mapsto \log_b y$ is the inverse of the function $x \mapsto b^x$; see Section 3.1).

Thus students should be comfortable with using inverse functions by the time they reach the inverse trigonometric functions (arccosine, arcsine, and arctangent) in Section 5.1. This familiarity with inverse functions should help students deal with inverse operations (such as antidifferentiation) when they reach calculus.

Chapter 2 (*Linear, Quadratic, Polynomial, and Rational Functions*) should be mostly review of what students learned in their intermediate algebra course. I placed the more demanding Chapter 1 first because there is a serious danger of boring students in a precalculus class if they develop an early feeling that they already know all this material.

Logarithms, e, and Exponential Growth

The base for logarithms in Chapter 3 is arbitrary, although most of the examples and motivation in the early part of Chapter 3 use logs base 2 or logs base 10.

Logarithms play a key role in calculus, but many calculus instructors complain that too many students lack appropriate algebraic manipulation skills with logarithms.

All precalculus textbooks present radioactive decay as an example of exponential decay. Amazingly, the typical precalculus textbook states that if a radioactive isotope has a half-life of h, then the amount left at time t will equal e^{-kt} times the amount at time 0, where $k = \frac{\ln 2}{h}$.

A much clearer formulation would state, as this textbook does, that the amount left at time t will equal $2^{-t/h}$ times the amount at time 0. The unnecessary use of e and $\ln 2$ in this context may suggest to students that e and natural logarithms have only contrived and artificial uses, which is not the message that students should receive from their textbook.

Similarly, many precalculus textbooks consider, for example, a colony of bacteria doubling in size every 3 hours, with the textbook then producing the formula $e^{(t \ln 2)/3}$ for the growth factor after t hours. The simpler and natural formula $2^{t/3}$ seems not to be mentioned in such books. This book presents the more natural approach to such issues of exponential growth and decay.

The crucial concepts of e and natural logarithms are introduced in the second half of Chapter 3. Most precalculus textbooks either present no motivation for e or motivate e via continuously compounding interest or through the limit of an indeterminate expression of the form 1^∞; these concepts are difficult for students at this level to understand.

About half of calculus (namely, integration) deals with area, but most precalculus textbooks barely mention the subject.

Chapter 3 presents a clean and well-motivated approach to e and the natural logarithm. This approach uses the area (intuitively defined) under the curve $y = \frac{1}{x}$, above the x-axis, and between the lines $x = 1$ and $x = c$.

A similar approach to e and the natural logarithm is common in calculus courses. However, this approach is not usually adopted in precalculus textbooks. Using obvious properties of area, the simple presentation given here shows how these ideas can come through clearly without the technicalities of calculus or the messy notation of Riemann sums. Indeed, this precalculus approach to the exponential function and the natural logarithm shows that a good understanding of these subjects need not wait until the calculus course. Students who have seen the approach given here should be well prepared to deal with these concepts in their calculus courses.

The approach taken here also has the advantage that it easily leads, as shown in Chapter 3, to the approximation $\ln(1 + h) \approx h$ for small values of h. Furthermore, the same methods show that if r is any number, then $\left(1 + \frac{r}{x}\right)^x \approx e^r$ for large values of x. A final bonus of this approach is that the connection between continuously compounding interest and e becomes a nice corollary of natural considerations concerning area.

Trigonometry

Trigonometry is the hardest part of precalculus for most students.

This book gives a gentle introduction to trigonometry, making sure that students are comfortable with the unit circle and with radians before defining the trigonometric functions.

Rather than following the practice of most precalculus books of defining six trigonometric functions all at once, this book has a section on the cosine and sine functions. Then the next section introduces the tangent function and finally the secant, cosecant, and cotangent functions. These latter three functions, which are simply the reciprocals of the three key trigonometric functions, add little content or understanding; thus they do not receive much attention here.

Should the trigonometric functions be introduced via the unit circle or via right triangles? Calculus requires the unit-circle approach because, for example, discussing the Taylor series for $\cos x$ requires us to consider negative values of x and values of x that are more than $\frac{\pi}{2}$ radians. Thus this textbook uses the unit-circle approach, but quickly gives applications to right triangles. The unit-circle approach also allows for a well-motivated introduction to radians.

Most precalculus textbooks define the trigonometric functions using four symbols: θ or t for the angle and $P(x, y)$ for the endpoint of the radius of the unit circle corresponding to that angle. Why is that endpoint usually called $P(x, y)$ instead of simply (x, y)? Even better than just dispensing with P, the symbols x and y can also be skipped by denoting the coordinates of the endpoint of the radius as $(\cos\theta, \sin\theta)$, thus defining the cosine and sine. The standard approach of defining $\cos\theta = x$ and $\sin\theta = y$ causes problems when students get to calculus and need to deal with $\cos x$. If students have memorized the notion that cosine is the x-coordinate, then they will be thinking that $\cos x$ is the x-coordinate of ... oops, this is two different uses of x. To avoid the confusion discussed above, this book uses only one symbol to define the trigonometric functions.

What's New in this Third Edition

- The chapter on systems of linear equations from the previous edition has been eliminated, as has the appendix on parametric curves. Both these items, which deal with topics that are not needed for first-semester calculus, are available as electronic supplements. They are also available in my *Algebra and Trigonometry* book.
- The section on transformations of trigonometric functions has been moved to Chapter 5.
- What are now Chapters 6 and 7 were in the reverse order in the previous edition. Chapter 7 has a new title.
- The main text font has been changed from Lucida to URW Palladio, which is a legal clone of Palatino. The math fonts have been changed from various versions of Lucida to various versions of URW Palladio, Pazo Math, and Computer Modern.

For more information on the typesetting of this book, see the Colophon at the end of the book.

- The paper length has been slightly expanded by three-eighths of an inch.
- The new fonts and new page size mean new page breaks and new line breaks. LaTeX handles line breaks well. However, I had to do extensive rewriting to make page breaks come out well. For example, students almost always have an entire Example visible without turning a page.

The content changes and format changes result in a book that is about one hundred pages shorter than the previous version.

- Each full page of text now contains at least two marginal notes, as compared to at least one marginal note in the previous edition. A figure or photo counts as a marginal note. When a figure or photo has a caption, the caption does not count as an additional marginal note. The word Example does not count as a marginal note.
- Eighteen new photos relevant to the content have been added.
- A new color scheme has been implemented. Definition boxes are now yellow and result boxes are now blue. Example lines are now orange, and example labels are now white inside orange.
- Definition boxes, result boxes, learning objectives boxes, and example label boxes now have rounded corners for a gentler look.
- Definition boxes and result boxes now have their titles in a darker-shaded sub-box for a catchy appearance.
- Numerous improvements have been made throughout the text based upon suggestions from faculty and students who used the previous edition.
- New exercises have been added in almost all sections. The Appendix now includes worked-out solutions to the Appendix's exercises.

Comments Welcome

I seek your help in making this a better book. Please send me your comments and your suggestions for improvements. Thanks!

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