Appendix B: Parametric Curves

WORKED-OUT SOLUTIONS *to Odd-Numbered Exercises*

Do not read these worked-out solutions before attempting to do the exercises yourself. Otherwise you may mimic the techniques shown here without understanding the ideas.

Best way to learn: Carefully read the section of the textbook, then do all the odd-numbered exercises and check your answers here. If you get stuck on an exercise, then look at the worked-out solution here.

For Exercises 1–4, consider the parametric curve described by the given ordered pair of functions defined on the given interval.

(a) ***What is the initial point of the parametric curve?***

(b) ***What is the endpoint of the parametric curve?***

(c) ***Sketch the parametric curve.***

(d) ***Is the parametric curve the graph of some function?***

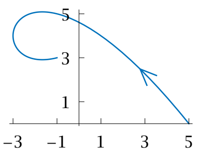
1 $(2t^2 - 8t + 5, t^3 - 6t^2 + 10t)$ for t in $[0, 3]$

SOLUTION

(a) Taking $t = 0$, we see that the initial point of this parametric curve is $(5, 0)$, as shown in the figure below.

(b) Taking $t = 3$, we see that the endpoint of this parametric curve is $(-1, 3)$, as shown in the figure below.

(c) If you are using *WolframAlpha*, see Example 2 for the kind of input that produces this figure.



(d) This curve fails the vertical line test, and thus it is not the graph of any function.

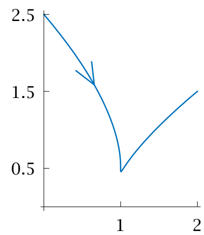
3 $\left(t^3 + 1, \dfrac{3t^2 - t + 1}{2}\right)$ for t in $[-1, 1]$

SOLUTION

(a) Taking $t = -1$, we see that the initial point of this parametric curve is $(0, \frac{5}{2})$, as shown in the figure below.

(b) Taking $t = 1$, we see that the endpoint of this parametric curve is $(2, \frac{3}{2})$, as shown in the figure below.

(c)



(d) This curve passes the vertical line test, and thus it is the graph of some function.

For Exercises 5–6, answer the following questions using the given information.

(a) ***Describe the path of the basketball as a parametric curve.***

(b) ***How long after the basketball is thrown does it hit the ground?***

(c) ***How far away from the thrower (in the horizontal direction) is the basketball when it hits the ground?***

(d) ***How high is the basketball at its maximum height?***

5 A basketball is thrown from height 5 feet with initial horizontal velocity 20 feet per second and initial vertical velocity 15 feet per second.

SOLUTION

(a) The parametric curve described by

$$(20t, -16.1t^2 + 15t + 5)$$

gives the path of the basketball. Here we are assuming that the basketball is thrown at time $t = 0$. The first coordinate $20t$ is the horizontal distance of the basketball from the thrower at time t seconds. The second coordinate $-16.1t^2 + 15t + 5$ is the height of the basketball at time t seconds. Part (b) shows that this description of the basketball's path is valid only for t in the interval $[0, 1.19]$.

(b) The basketball hits the ground when $-16.1t^2 + 15t + 5 = 0$. The quadratic formula shows that this happens when $t \approx 1.19$ seconds. Thus the formula for the parametric curve given in part (a) is valid only for t in the interval $[0, 1.19]$.

(c) When the basketball hits the ground at time 1.19, its horizontal distance from the thrower is 20×1.19 feet, which is 23.8 feet.

(d) To find the maximum height of the basketball, we complete the square:

$$\begin{aligned}&-16.1t^2 + 15t + 5\\ &\qquad = -16.1\big[t^2 - \tfrac{15}{16.1}t\big] + 5\\ &\qquad = -16.1\big[\big(t - \tfrac{15}{32.2}\big)^2 - \big(\tfrac{15}{32.2}\big)^2\big] + 5\\ &\qquad = -16.1\big(t - \tfrac{15}{32.2}\big)^2 + \tfrac{15^2}{64.4} + 5\end{aligned}$$

The expression above shows that the maximum height of the basketball is attained when $t = \frac{15}{32.2}$ and that the basketball's height at that time will be $\frac{15^2}{64.4} + 5$ feet, which is approximately 8.5 feet.

For Exercises 7–8, answer the following questions using the given information.

(a) ***Describe the path of the basketball as a parametric curve.***

(b) ***How long after the basketball is thrown does it hit the wall?***

(c) ***How high is the basketball when it hits the wall?***

7 A basketball is thrown at a wall 40 feet away from height 6 feet with initial horizontal velocity 35 feet per second and initial vertical velocity 20 feet per second.

SOLUTION

(a) The parametric curve $(35t, -16.1t^2 + 20t + 6)$ describes the path of the basketball. Here we are assuming that the basketball is thrown at time $t = 0$. The first coordinate $35t$ is the horizontal distance of the basketball from the thrower at time t seconds. The second coordinate $-16.1t^2 + 20t + 6$ is the height of the basketball at time t seconds. Part (b) shows that this description of the basketball's path is valid only for t in the interval $[0, 1.14]$.

(b) The basketball hits the wall when $35t = 40$, which means $t = \frac{40}{35} \approx 1.14$ seconds. Thus the formula for the parametric curve given in part (a) is valid only for t in the interval $[0, 1.14]$.

(c) When the basketball hits the wall at time 1.14, its height is $-16.1 \times 1.14^2 + 20 \times 1.14 + 6$ feet, which is approximately 7.9 feet.

9 Find a function f such that the parametric curve described by $(t - 1, t^2)$ for t in $[0, 2]$ is the graph of f.

SOLUTION Let $s = t - 1$. As t varies over the interval $[0, 2]$, clearly s varies over the interval $[-1, 1]$.

Because $t = s + 1$, we have

$$(t - 1, t^2) = (s, (s + 1)^2).$$

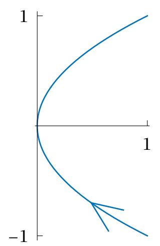
Thus if we define a function f with domain $[-1, 1]$ by $f(s) = (s + 1)^2$, then the graph of f is the parametric curve described by $(t - 1, t^2)$ for t in $[0, 2]$.

11 (a) Sketch the parametric curve described by (t^2, t) for t in the interval $[-1, 1]$.

(b) Find a function f such that the parametric curve in part (a) could be obtained by flipping the graph of f across the line with slope 1 that goes through the origin.

SOLUTION

(a)

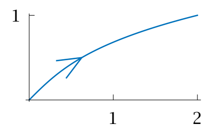


(b) Flipping a parametric curve across the line with slope 1 that goes through the origin interchanges the coordinates. Thus we are looking for a function f whose graph is the parametric curve described by (t, t^2) for t in $[-1, 1]$. Thus we define f by $f(t) = t^2$, with the domain of f equal to $[-1, 1]$.

13 Use a parametric curve to sketch the inverse of the function f defined by $f(x) = x^3 + x$ on the interval $[0, 1]$.

SOLUTION

The graph of f is the parametric curve described by $(t, t^3 + t)$ for t in $[0, 1]$. Thus the graph of f^{-1} is the parametric curve described by $(t^3 + t, t)$ for t in $[0, 1]$, as shown here.



15 Find the equation of the ellipse in the xy-plane given by the parametric curve described by $(2\cos t, 7\sin t)$ for t in $[0, 2\pi]$.

SOLUTION Note that

$$\frac{(2\cos t)^2}{4} + \frac{(7\sin t)^2}{49} = \frac{4\cos^2 t}{4} + \frac{49\sin^2 t}{49}$$
$$= \cos^2 t + \sin^2 t$$
$$= 1.$$

Thus setting $x = 2\cos t$ and $y = 7\sin t$, we have the equation of the ellipse $\frac{x^2}{4} + \frac{y^2}{49} = 1$.

17 Write the ellipse $\frac{x^2}{16} + \frac{y^2}{81} = 1$ as a parametric curve.

SOLUTION The parametric curve described by $(4\cos t, 9\sin t)$ for t in $[0, 2\pi)$ gives the desired ellipse. There are also many other correct solutions, such as the parametric curve described by $(4\sin t, 9\cos t)$ for t in $(-\pi, \pi]$.

19 Write the ellipse $7x^2 + 5y^2 = 3$ as a parametric curve.

SOLUTION The equation $7x^2 + 5y^2 = 3$ can be rewritten as

$$\left(\sqrt{\tfrac{7}{3}}x\right)^2 + \left(\sqrt{\tfrac{5}{3}}y\right)^2 = 1.$$

The equation above shows that a good way to write this ellipse as a parametric curve is to take

$$\cos t = \sqrt{\tfrac{7}{3}}x \quad \text{and} \quad \sin t = \sqrt{\tfrac{5}{3}}y.$$

Solving these equations for x and y leads to the parametric curve described by

$$\left(\sqrt{\tfrac{3}{7}}\cos t, \sqrt{\tfrac{3}{5}}\sin t\right)$$

for t in $[0, 2\pi)$. There are also many other correct solutions.

For Exercises 21–36, explain how the parametric curve described by the given ordered pair of functions could be obtained from the parametric curve shown in Example 2; assume t is in $[0, 4]$ in all cases. Then sketch the given parametric curve.

The coordinates in Exercises 21–36 can be easily obtained from the coordinates

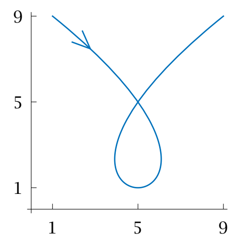
$$(t^3 - 6t^2 + 10t, 2t^2 - 8t + 9)$$

in Example 2. The intention here is that you shift, stretch, or flip the parametric curve in Example 2 to obtain your results. Do not use a computer or graphing calculator to sketch the graphs in these exercises, because doing so would deprive you of a good understanding of transformations of parametric curves.

21 $(t^3 - 6t^2 + 10t + 1, 2t^2 - 8t + 9)$

SOLUTION These coordinates are obtained by adding 1 to the first coordinate of every point on the parametric curve from Example 2.

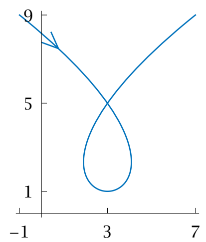
Thus the parametric curve with these coordinates is obtained by shifting the parametric curve from Example 2 right 1 unit, giving the curve shown here.



23 $(t^3 - 6t^2 + 10t - 1, 2t^2 - 8t + 9)$

SOLUTION These coordinates are obtained by subtracting 1 from the first coordinate of every point on the parametric curve from Example 2.

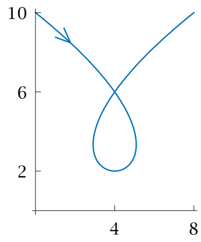
Thus the parametric curve with these coordinates is obtained by shifting the parametric curve from Example 2 left 1 unit, giving the curve shown here.



25 $(t^3 - 6t^2 + 10t, 2t^2 - 8t + 10)$

SOLUTION These coordinates are obtained by adding 1 to the second coordinate of every point on the parametric curve from Example 2.

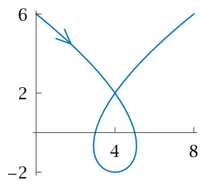
Thus the parametric curve with these coordinates is obtained by shifting the parametric curve from Example 2 up 1 unit, giving the curve shown here.



27 $(t^3 - 6t^2 + 10t, 2t^2 - 8t + 6)$

SOLUTION These coordinates are obtained by subtracting 3 from the second coordinate of every point on the parametric curve from Example 2.

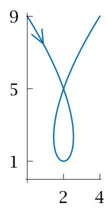
Thus the parametric curve with these coordinates is obtained by shifting the parametric curve from Example 2 down 3 units, giving the curve shown here.



29 $\left(\dfrac{t^3 - 6t^2 + 10t}{2}, 2t^2 - 8t + 9\right)$

SOLUTION These coordinates are obtained by multiplying the first coordinate of every point on the parametric curve from Example 2 by $\frac{1}{2}$.

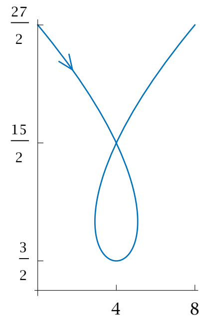
Thus the parametric curve with these coordinates is obtained by horizontally stretching the parametric curve from Example 2 by a factor of $\frac{1}{2}$, giving the curve shown here.



31 $(t^3 - 6t^2 + 10t, \frac{3}{2}(2t^2 - 8t + 9))$

SOLUTION These coordinates are obtained by multiplying the second coordinate of every point on the parametric curve from Example 2 by $\frac{3}{2}$.

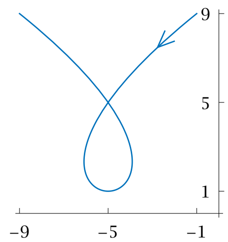
Thus the parametric curve with these coordinates is obtained by vertically stretching the parametric curve from Example 2 by a factor of $\frac{3}{2}$, giving the curve shown here.



33 $(-(t^3 - 6t^2 + 10t + 1), 2t^2 - 8t + 9)$

SOLUTION These coordinates are obtained by multiplying the first coordinate of every point on the parametric curve from Exercise 21 by -1.

Thus the parametric curve with these coordinates is obtained by flipping the parametric curve from Exercise 21 across the vertical axis, giving the curve shown here.



35 $(t^3 - 6t^2 + 10t, -(2t^2 - 8t + 6))$

SOLUTION These coordinates are obtained by multiplying the second coordinate of every point on the parametric curve from Exercise 27 by -1.

Thus the parametric curve with these coordinates is obtained by flipping the parametric curve from Exercise 27 across the horizontal axis, giving the curve shown here.